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CONTINUED FRACTION EVALUATION OF THE STUMPFF FUNCTIONS OF SPACE DYNAMICS

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Abstract

Abstract .In this paper, continued fraction expansion of the Stumpff functions of space dynamics are developed using Euler theory. An efficient and new computational algorithm based on this expansion is developed . Numerical results of the algorithm are also given.

1. Introduction

A complete interplanetary space mission consists essentially of [1

] three parts: (1) Escape from the departure planet,

(2) Heliocentric transfer,

(3) Capture by the target planet.

The first and the last parts involve hyperbolic orbits, whereas the intermediate portion, commonly depicted as a heliocentric ellipse, may also be heliocentric parabola or hyperbola. Thus far we have been obliged to use different formulations [2] to describe the motion in each of these various orbits. But, as we just mentioned, during the space mission all types of orbits appear, and in addition, which is the most critical, in some systems the type of an orbit is occasionally changed by perturbing forces acting during interval of time [3]. This problem, leads to drastic situations, as for example, one may use set of elliptic formulations (say) to determine orbital parameters of hyperbolic or parabolic orbits, which in turn leads to inaccurate predictions.

Consequently, universal formulations are desperately needed so that, orbit predictions will be free of the troubles which arise with the classical equations when transition from one type of orbit to another occurs.

Universal formulations, that are the formulations simultaneously valid for any type of orbits, could be developed using new family of transcendental functions [2]. Of these are Stumpff functions defined by Equation (2.1).

There are several methods for the evaluation of Stumpff functions, all depending on polynomial evaluations [4] and [5]. In fact, continued fraction expansions are generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series.

Due to the importance of accurate evaluation of Stumpff functions for orbital predictions, and on the other hand, due to the efficiency of continued fraction for evaluating functions are what motivated our work: to establish computational algorithm for the Stumpff functions based on their continued fraction expansion.

2- Basic Formulations

2.1. STUMPFF FUNCTIONS

The stumpff functions are family of transcendental functions, defined [4] by

$$C_n(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(2k+n)!} \; ; \; \forall n \geq 0. \tag{2.1}$$

They are well defined for a complex variable z, since the power series is convergent at any point of the complex plane. They are real valued for real z. Note also that,

$$C_n(o) = \frac{1}{n!}. \tag{2.2}$$

The relations between $C_n(z)$ and the trigonometric and hyperbolic functions could be generated from

$$C_{2n}(\Psi) = \begin{cases} \dfrac{(-1)^n}{\left(\sqrt{\Psi}\right)^{2n}} \left[\operatorname{Cos}\sqrt{\Psi} - \operatorname{Cos}\sqrt{\Psi}\,\Big|_{2(n-1)}\right] & \text{if } \Psi \succ 0, \\ \\ \dfrac{1}{\left(\sqrt{-\Psi}\right)^{2n}} \left[\operatorname{Cosh}\sqrt{-\Psi} - \operatorname{Cosh}\sqrt{-\Psi}\,\Big|_{2(n-1)}\right] & \text{if } \Psi \prec 0, \end{cases} \tag{2.3}$$

$$C_{2n+1}(\Psi) = \begin{cases} \dfrac{(-1)^n}{\left(\sqrt{\Psi}\right)^{2n+1}} \left[\operatorname{Sin}\sqrt{\Psi} - \operatorname{Sin}\sqrt{\Psi}\,\Big|_{2n-1}\right] & \text{if } \Psi \succ 0, \\ \\ \dfrac{1}{\left(\sqrt{-\Psi}\right)^{2n+1}} \left[\operatorname{Sinh}\sqrt{-\Psi} - \operatorname{Sinh}\sqrt{-\Psi}\,\Big|_{2n-1}\right] & \text{if } \Psi \prec 0, \end{cases} \tag{2.4}$$

where $\mathbf{f(w)}\rfloor_m$ is the polynomial obtained by truncating the power series expansion of f(w) after the mth power. As for examples,

$$\mathbf{Cos}\sqrt{\Psi}\,\big|_{-2} = \mathbf{Cosh}\sqrt{-\Psi}\,\big|_{-2} = 0 \;;\qquad \mathbf{Cos}\sqrt{\Psi}\,\big|_{0} = \mathbf{Cosh}\sqrt{-\Psi}\,\big|_{0} = 1\;;$$

$$\mathbf{Sin}\sqrt{\Psi}\,\big|_{-1} = \mathbf{Sinh}\sqrt{-\Psi}\,\big|_{-1} = 0 \;;\quad \mathbf{Sin}\sqrt{\Psi}\,\big|_{1} = \sqrt{\Psi}\;;\mathbf{Sinh}\sqrt{-\Psi}\,\big|_{1} = \sqrt{-\Psi}\,.$$

2.2. EULER'S TRANSFORMATION

An infinite series of functions could be converted into a continued fraction according to Euler's transformation [2] which is

$$\sum_{k=o}^{\infty} \mathbf{U}_k \equiv \cfrac{\mathbf{n}_1}{\mathbf{d}_1 + \cfrac{\mathbf{n}_2}{\mathbf{d}_2 + \cfrac{\mathbf{n}_3}{\mathbf{d}_3 + \cfrac{\mathbf{n}_4}{\vdots}}}} \equiv \frac{\mathbf{n}_1}{\mathbf{d}_1 +}\,\frac{\mathbf{n}_2}{\mathbf{d}_2 +}\,\frac{\mathbf{n}_3}{\mathbf{d}_3 +}\frac{\mathbf{n}_4}{\mathbf{d}_4 +}\cdots\cdots \qquad (2.5)$$

where

$$\mathbf{n}_1 = \mathbf{U}_0\;;\quad \mathbf{n}_2 = \mathbf{U}_1\;;\;\mathbf{n}_i = \mathbf{U}_{i-1} * \mathbf{U}_{i-3} \quad \forall \mathbf{i} \geq 3\,, \qquad (2.6)$$

$$\mathbf{d}_1 = 1\;;\;\mathbf{d}_j = \mathbf{U}_{j-2} - \mathbf{U}_{j-1}\;\forall \mathbf{j} \geq 1 \qquad (2.7)$$

3. Computational Developments

3.1.TOP- DOWN CONTINUED FRACTION EVALUATION

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the n'th convergent were accumulated separately with three – term recurrence formulae. The drawback to the first method is, obviously, having to decide far down the fraction to begin in order to ensure convergence. The drawback to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm which works from top down while avoiding numerical difficulties would be ideal from a programming standpoint. Goutschi [6] proposed very concise algorithm to evaluate continued fraction from the top down may be summarized as follows. If the continued fraction is given as

$$C \equiv \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \cfrac{n_4}{\vdots}}}} \equiv \frac{n_1}{d_1 +}\,\frac{n_2}{d_2 +}\,\frac{n_3}{d_3 +}\,\frac{n_4}{d_4 +}\cdots\cdots$$

then initialize the following parameters

$$a_1 = 1$$

$$b1 = n_1/d_1$$

$$c_1 = n_1/d_1$$

and iterate (k=1,2,....) according to

$$a_{k+1} = \frac{1}{1 + \left[\frac{n_{k+1}}{d_k\ d_{k+1}}\right] a_k},$$

$$b_{k+1} = [a_{k+1} - 1]\ b_k,$$

$$c_{k+1} = c_k + b_{k+1}.$$

In the limit, the c sequence converges to the value of continued fraction.

However, the complexity of the n's and d's coefficients [of Equations (2.6) and (2.7)] with u's given by Equation (2.1) make it somewhat inefficient to use Gautschi's method directly.

Instead, with the introduction of only five parameters α, β, γ d_o and d_g and the recurrence relation for the series terms of

Stumpff function $C_M(\chi)$

$$\beta = -\frac{\chi \ \mathrm{Sign}\ (\chi)}{(2m+M)(2m+M-1)}\ \beta, \tag{3.1}$$

where

$$\mathrm{Sign}(x) = \begin{cases} 1 & \text{if } x \succ 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x \prec 0 \end{cases} \tag{3.2}$$

to relate the mth computational stage to the previous one, Gautschi's method with Euler transformation may be reformatted into a form more suitable for digital computers. It should be mentioned that, the argument χ of the Stumpff functions for the space dynamics applications is always rational such that

$$\chi \in \overline{\chi} = Q - z = \left\{ \chi : \chi = \frac{p}{q} \ , \quad q \neq 0 \quad , \ 1 \right\}$$

The method proceeds as in the following computational algorithm.

3.2. COMPUTATIONAL ALGORITHM

- **Purpose :** To compute the value of the Stumpff function $C_M(\chi)$ for given χ and M, where $\chi \in \bar{\chi}$ and M non—negative integer, using continued fraction.

- **Input :** χ, M, $\in$: Tolerance specifies the upper bound of the absolute value of b[in Gautschi's method of Subsection 3.1] and below which the calculations are terminated, N: The maximum number of recurrent calculations.

- **Output :** $C \equiv C_M(\chi)$, IER : Resulting error parameter coded as follows.

  * IER = 0, means that, the convergence in computing C is achieved in number of cycles ≤N within accuracy specified by $\in$.

  * IER = 1, means that, no convergence after N cycles is achieved within the accuracy specified by $\in$.

- **Computational sequence**

  1. set IER= 0

2. $\alpha = \dfrac{1}{M!}$; *If* $\chi = 0,\ C = \alpha,$ *go to step* 7 ; $\beta = \dfrac{\chi * Sign(\chi)}{(M+2)(M+1)}\alpha$; $d_o = 1$

$a = 1;\ \ b = \alpha$; $C = \alpha$; $\gamma = 1$; $m = 1,$

3. $n = \beta * \gamma$; $d_g = \alpha - \beta$; $a = \dfrac{1}{1 + \left(\dfrac{n}{d_o + d_g}\right) * a}$; $b = (a-1) * b$; $C = C + b.$

4. *IF* $|b| \prec \in$, *go to setp* 7.

5. $m = m + 1$; *IF* $m \succ N$, *set* $IER = 1$, *go to step* 7.

6. $B = 2m + M$; $\gamma = \alpha$; $\alpha = \beta$; $\beta = \dfrac{Sign(\chi) * \chi}{B * (B-1)}\beta$; $d_o = d_g$ *go to step* 3.

7. *End.*

According to the adopted range of the argument χ of Equation (3.1), it is thus clear that, $\mathbf{d_g}$ of step 3 in the above computational algorithm, is never equal to zero, hence the algorithm is free from any singularity.

3.3 NUMERICAL RESULTS

FORTRAN 77 Code was constructed for digital computations of the above algorithm with N=15 and $\in = 10^{-12}$. The numerical results are listed in Table I for

$\mathbf{C_M(\chi)}$; $\chi = -4.5\ (1)\ (4.5)$ **and** $\mathbf{M} = 0\ (1)\ 11.$

In concluding the present paper, an efficient and new computational algorithm for the Stumpff functions of space dynamics was established using continued fraction expansion.

TABLE I

Values of The Stumpff Functions CM(χ)
And M=0 (1) 11 For χ=-4.5 (1) 4.5

X	C0(X)	C1(X)	C2(X)
-.4500000000000D+01	.4231008983290D+01	.1938007970210D+01	.7180019962867D+00
-.3500000000000D+01	.3323835714773D+01	.1694350601162D+01	.6639530613637D+00
-.2500000000000D+01	.2533114295765D+01	.1471960330364D+01	.6132457183061D+00
-.1500000000000D+01	.1848565174647D+01	.1269433785832D+01	.5657101164312D+00
-.5000000000000D+00	.1260591836521D+01	.1085441641273D+01	.5211836730427D+00
.0000000000000D+00	.1000000000000D+01	.1000000000000D+01	.5000000000000D+00
.5000000000000D+00	.7602445970756D+00	.9187253698656D+00	.4795108058487D+00
.1500000000000D+01	.3391859889869D+00	.7680941196125D+00	.4405426740087D+00
.2500000000000D+01	-.1034231890521D-01	.6324217062814D+00	.4041369275621D+00
.3500000000000D+01	-.2955511274930D+00	.5106437200905D+00	.3701574649980D+00
.4500000000000D+01	-.5231338942889D+00	.4017547423895D+00	.3384741987309D+00

TABLE I (CONTINUED)

X	C3(X)	C4(X)	C5(X)
-.4500000000000D+01	.2084462156022D+00	.4844488806372D-01	.9284344207885D-02
-.3500000000000D+01	.1983858860462D+00	.4684373181820D-01	.9062634108442D-02
-.2500000000000D+01	.1887841321454D+00	.4529828732244D-01	.8846986191508D-02
-.1500000000000D+01	.1796225238879D+00	.4380674428746D-01	.8637238147472D-02
-.5000000000000D+00	.1708832825452D+00	.4236734608542D-01	.8433231757095D-02
.0000000000000D+00	.1666666666667D+00	.4166666666667D-01	.8333333333333D-02
.5000000000000D+00	.1625492602689D+00	.4097838830252D-01	.8234812795607D-02
.1500000000000D+01	.1546039202583D+00	.3963821732753D-01	.8041830938888D-02
.2500000000000D+01	.1470313174874D+00	.3834522897517D-01	.7854139671689D-02
.3500000000000D+01	.1398160799741D+00	.3709786714343D-01	.7671596197866D-02
.4500000000000D+01	.1329433905801D+00	.3589462250425D-01	.7494061352570D-02

TABLE I (Continued)

X	C6(X)	C7(X)	C8(X)
-.4500000000000D+01	.1506271421567D-02	.2113357499004D-03	.2608500726177D-04
-.3500000000000D+01	.1479161471866D-02	.2083716500310D-03	.2579216656495D-04
-.2500000000000D+01	.1452648262309D-02	.2054611432700D-03	.2550374936804D-04
-.1500000000000D+01	.1426718413865D-02	.2026032094257D-03	.2521968331719D-04
-.5000000000000D+00	.1401358837516D-02	.1997968475227D-03	.2493989725347D-04
.0000000000000D+00	.1388888888889D-02	.1984126984127D-03	.2480158730159D-04
.5000000000000D+00	.1376556728292D-02	.1970410754525D-03	.2466432119353D-04
.1500000000000D+01	.1352299559423D-02	.1943349296304D-03	.2439288631056D-04
.2500000000000D+01	.1328575076600D-02	.1916774646577D-03	.2412552491554D-04
.3500000000000D+01	.1305371292353D-02	.1890677529906D-03	.2386217043876D-04
.4500000000000D+01	.1282676480536D-02	.1865048846141D-03	.2360275741172D-04

TABLE I (CONTINUED)

X	C9(X)	C10(X)	C11(X)
-.4500000000000D+01	.2871789219496D-05	.2852044355959D-06	.2579051046611D-07
-.3500000000000D+01	.2845414748080D-05	.2830226466762D-06	.2562366448049D-07
-.2500000000000D+01	.2819377942925D-05	.2808648265818D-06	.2545840821075D-07
-.1500000000000D+01	.2793674008676D-05	.2787306770668D-06	.2529472418527D-07
-.5000000000000D+00	.2768298219966D-05	.2766199037598D-06	.2513259513567D-07
.0000000000000D+00	.2755731922399D-05	.2755731922399D-06	.2505210838544D-07
.5000000000000D+00	.2743245920401D-05	.2745322161123D-06	.2497200399443D-07
.1500000000000D+01	.2718512521560D-05	.2724673273495D-06	.2481293389253D-07
.2500000000000D+01	.2694093502006D-05	.2704249544209D-06	.2465536815705D-07
.3500000000000D+01	.2669984406317D-05	.2684048179511D-06	.2449929030889D-07
.4500000000000D+01	.2646180844126D-05	.2664066421923D-06	.2434468406047D-07

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تقييم دوال استنف لديناميكا الفضاء بطريقة الكسر المستمر

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