Mansoura University	2013	Time : 180 min
Faculty of Engineering		Real Analysis
Department of Engg. Math. and Phys.		M. Sc. Exam

[1]-(a) If $X = (x_n)$ and $Y = (y_n)$ are convergent sequences real numbers and if $x_n \leq y_n$ for all $n \in N$ then prove that

$$\lim (x_n) \leq \lim (x_n)$$

(b) If $X = (x_n)$ is a convergent sequence of real numbers and i $x_n \geq 0$ for all $n \in N$ then prove that

$$x = \lim (x_n) \geq 0.$$

(c) Let (x_n) be a sequence of positive real numbers such that $L = \lim \left(\frac{x_{n+1}}{x_n}\right)$ exists. Prove that, if $L < 1$, then (x_n) converges and $\lim (x_n) = 0$.

(d) Use the Squeeze Theorem to determine the limit

$$\left((n)^{1/n^2}\right)$$

(e) Prove that every contractive sequence is a cauchy sequence and convergent.

[2]-(a) If $A \subseteq \mathbb{R}$, and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$ then prove that f is bounded on some neighborhood of c.

(b) Show that if $c > 1$, then the following series is convergent

$$\sum \frac{1}{n\,(\ln n)(\ln \ln n)^c}$$

(c) Suppose that $\lim_{x\to c} f(x) = L$ where $L > 0$ and that $\lim_{x\to c} g(x) = \infty$. Show that $\lim_{x\to c} f(x)g(x) = \infty$. If $L = 0$, show by example that this conclusion may fail.

(d) If $f : A \to \mathbb{R}$ is a Lipschitz function, prove that f is uniformly continuous on A.

[3]-(a) Discuss the convergence or the divergence of the series with nth term

$$1.\ (\ln n)^{-\ln n}, \qquad 2.\ (\ln n)\, e^{-\sqrt{n}}$$

(b) Suppose that $\sum a_n$ is a convergent series of real numbers. Either prove that $\sum b_n$ converges or given a counter-example, when we define b_n by

$$(1).\ a_n \sin n, \qquad (2).\ \frac{a_n}{1+|a_n|}$$

(c) Prove that a Cauchy sequence of real numbers is bounded? Give an example of a bounded sequence that is not a Cauchy sequence.

[4]-(a) Show that if $f_1, \ldots, f_n$ are in $\mathbb{R}[a, b]$ and if $k_1, \ldots, k_n \in \mathbb{R}$, then the linear combination

$$f = \sum_{i=1}^{n} k_i f_i$$

belongs to $\mathbb{R}[a, b]$ and

$$\int_a^b f = \sum_{i=1}^{n} k_i \int_a^b f_i.$$

(b) Let f be defined on $[0, 2]$ by

$$f(x) = \begin{cases} 1, & x \neq 1; \\ 0, & x = 1. \end{cases}$$

Show that the Darboux integral exists and find its value.

Assoc. Prof. Dr. El-Gamel