*Sci. J. Fac. Sci. Menoufia Univ. Vol. VIII(1994)*

# ON HYPERPLANES IN BANACH SPACES

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ABSTRACT.

*In this paper we shall prove the following theorem:"Let $E$ be a Banach space and $F \subset E$ be a closed subspace of $E$ such that $\operatorname{codim} F = \dim (E/F) = k < \infty$, and let $g$ be a strongly non-singular form on $E$ such that $g|_F$ is weak non singular. Then there exists a $k$-dimensional vector subspace $G$ orthogonal complement to $F$ with respect to $g$ such that $g|_G$ is a weak non-singular form."*

## 1.Definitions.

1.1. By $L_n(E; \mathbb{R})$ we denote the space of all n-linear continuous functionals $:E^n \to \mathbb{R}$, where $E$ is a Banach space and $\mathbb{R}$ is the set of real numbers, $n$ is a positive integer

1.2. The bilinear form $g \in L_2(E; \mathbb{R})$ is said to be strongly non-singular ([1]) if

(i) $g$ is symmetric (i.e. $g(x,y) = g(y,x)$ for every $x,y \in E$)

(ii) $g$ associates a mapping

$$g^* : x \in E \to g^*_x = g(x,.) \in L(E; \mathbb{R}) = E^*$$

which is bijective.

$g$ is said to be a weak non-singular if we replaced (ii) by :

(ii)* If $g(x, y) = 0$ for every $y \in E$ , then $x = 0$

1.3. Let X be a vector space. A linear set $H \subset X$ is called a hypersubspace of X if $H \neq X$ and there exists a vector $y \in X$ such that X is the linear span of H and y ([2]), denoted by <H, y>. A hyperplane in X is any set of the form $x + H$ where $x \in X$ and H is a hypersubspace in X.([2]).

**Theorem**

Let E be a Banach space and $F \subset E$ be a closed subspace of E such that codim F = dim (E/F) = $k < \infty$, and let g be strongly non-singular form on E such that $g|_F$ is weak non singular. Then there exists a k-dimensional vector subspace G orthogonal complement to F with respect to g such that $g|_G$ is a weak non-singular form."

Proof

The proof of the theorem is a direct conclusion of the following three lemmas:

**Lemma 1.**

If F is a closed vector subspace of the Banach space E such that codim F = $k < \infty$, then there exist k-linear forms

$\alpha_1, \ldots, \alpha_k \in \mathrm{L}(E; \mathbb{R})$ such that $\bigcap_{i=1}^{k} \alpha_i^{-1}(o) = F$.

Proof.

Let $\alpha_1, \ldots, \alpha_k$ be the basis of the space E/F. Consider $e_1, \ldots, e_k \in E$ such that $P(e_i) = a_i$ , $i = 1, 2, \ldots, k$, where $P : E \to E/F$ is the canonical projection.

Let $\mathrm{L}(e_1, \ldots, e_k) = \{ \lambda^i e_i \ , \ \lambda^i \in \mathbb{R}, \ \ i = 1, \ldots, k\} = G$. Then we have the direct sum $E = G \oplus F$, i.e. for every $x \in E$,

$x = y + z$ where $y \in G$, $z \in F$, and this representation is unique.

To prove this fact, let $x \in E$, $P(x) = \lambda^i a_i$, $i = 1,\ldots,k$, $y = \lambda^j e_j \in G$, $j = 1,\ldots,k$, and $z = x - y$. Then $P(z) = P(x) - P(y) = 0$, and this implies that $z \in F$. To prove the uniqueness, if $0 = \lambda^j e_j + z$ and $z \in F$, then $0 = P(\lambda^j e_j) + P(z)$, i.e. $P(\lambda^j e_j) = 0$ which leads to $\lambda^j e_j = 0$, and then $\lambda^j = 0$, $j = 1,\ldots,k$. It follows that $z = 0$ i.e. if $0 = y + z$, $y \in G$, $z \in F$, then $y = z = 0$.

Since G is k-dimensional subspace of E, then it is closed. Therefore $E = G \oplus F$ is a direct sum of closed subspaces, which implies that the projections $P_G : E \to G$ and $P_F : E \to F$ are continuous.

Define the functionals $f_i : G \to \mathbb{R}$, $i = 1,\ldots,k$, where $f_i(e_j) = \delta_{ij}$ $\left(\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}\right)$, $i,j = 1,\ldots,k$. Taking into account that G is a k-dimensional topological linear subspace, then $f_i$, $i = 1,\ldots,k$ are continuous. Now assuming $\alpha_i = f_i \circ P_G$, then $\alpha_i$, $i = 1,\ldots,k$ are continuous. Since $P_G|_F = 0$, then $\alpha_i(F) = 0$, therefore for every $i = 1, 2, \ldots, k$

$$F \subseteq \alpha_i^{-1}(0) \tag{1.1}$$

Conversely, let $x \in \bigcap_{i=1}^{k} \alpha_i^{-1}(o)$, then $\alpha_1(x) = \ldots = \alpha_k(x) = 0$. It follows that if $x = \lambda^j e_j + z$, $j = 1,\ldots,k$, $z \in F$ and $\alpha_s(x) = (f_s \circ P_G)(x) = f_s(\lambda^j e_j) = \lambda^s = 0$, $s = 1,\ldots,k$. i.e. $x = z \in F$, then

$$\bigcap_{i=1}^{k} \alpha_i^{-1}(o) \subseteq F. \qquad (1.2)$$

From (1.1), (1.2) we have that $F = \bigcap_{i=1}^{k} \alpha_i^{-1}(o)$.

**LEMMA 2.**

A subspace F of the Banach space E is a hyperplane in E iff there exists a linear functional $\alpha \in L(E; \mathbb{R})$, $\alpha \neq 0$ such that $F = \alpha^{-1}(0)$.

**Proof.**

Let F be a hyperplane in E, then there exists a vector $x \in E$, $x \neq 0$ such that $F \oplus \langle x \rangle = E$, then from lemma 1 when k=1 there exists $\alpha \in L(E; \mathbb{R})$ such that $F = \alpha^{-1}(0)$.

Conversely, let $\alpha \in L(E; \mathbb{R})$, $\alpha \neq 0$ such that $F = \alpha^{-1}(0)$ then we have that

(i) $F \subset E$, $F \neq E$, since $\alpha \neq 0$,

(ii) F is a linear subspace of E,

(iii) F is closed, since $\alpha$ is continuous and $\{0\}$ is a closed subspace of $\mathbb{R}$. then $F = \alpha^{-1}(0)$ is closed in E.

Moreover, since $\alpha \neq 0$ then there exists a vector $x \in E$ such that $\alpha(x) \neq 0$. Let $\alpha(x) = \mu \neq 0$ and $x_0 = x/\mu$,

then $\alpha(x_0) = 1$.

Now we prove that $E = F \oplus \langle x_0 \rangle$. Let $z \in F \cap \langle x_0 \rangle$. then $\alpha(z) = \alpha(\lambda x_0) = \lambda \alpha(x_0) = \lambda = 0$, $\lambda \in \mathbb{R}$, then $z = 0$.

To prove that $E \subset F \oplus \langle x_0 \rangle$,

let $z \in E$, $z = (z - \alpha(z).x_0) + \alpha(z).x_0$, where $\alpha(z).x_0 \in \langle x_0 \rangle$. since $\alpha(z - \alpha(z).x_0) = 0$, then $z - \alpha(z).x_0 \in F$. Hence $F = \alpha^{-1}(0)$ is a hyperplane in E.

**Lemma 3.**

Let $g \in L_2(E; \mathbb{R})$ be a symmetric, strongly non-singular bilinear form. Then for every $x \in E$, $x \neq 0$,

$F = \{y \in E : g(x,y)=0\} = \langle x\rangle^{\perp}$ is a hyperplane in E. Conversely, if a hyperplane $F \subset E$ is given, then there exists a vector $x \in E$, $x \neq 0$ such that $F = \langle x\rangle^{\perp}$.

**Proof.**

If $x \in E$, $x \neq 0$, we we take $\alpha(x) \in L(E; \mathbb{R})$ such that for every $y \in E$, $\alpha_x(y) = g(x, y)$. Then for every

$y \in \langle x\rangle^{\perp}$, $\alpha_x(y) = 0$. i.e. $\langle x\rangle^{\perp} = \alpha^{-1}(0)$, therefore from lemma 1 we have $\langle x\rangle^{\perp}$ is a hyperplane in E.

Now, if F is a hyperplane in E then by lemma 2 there exists a linear form $\alpha \in L(E; \mathbb{R})$ such that $F = \alpha^{-1}(0)$. But since g is a strongly non-singular, then there exists $x \in E$ such that for every $y \in E$, $\alpha_x(y) = g(x, y)$. Hence

$F = \alpha_x^{-1}(0) = \{y \in E : \alpha_x(y) = g(x,y) = 0\} = \langle x\rangle^{\perp}$ .

Now we are ready to prove the considered theorem. Suppose E is a Banach space and $F \subset E$ is a closed subspace of E with codim $F = k < \infty$. Then by lemmas 1, 2 and 3, F can be represented in the form

$F = \bigcap_{i=1}^{k} \langle x_i\rangle^{\perp}$ such that for every $i = 1,\dots,k$, $\langle x_i\rangle^{\perp}$ is a hyperplane in E. we define a set G where

$\dim G = k < \infty$, $G = \langle x_1,\dots,x_k\rangle$. Now we have to show that

$E = G \oplus F$

Suppose that $z_0 \in G \cap F$, then $z_0 \in G$ and $z_0 \in F$. But from the definition of G. $z_0 \in F^{\perp}$, then $z_0 = 0$. therefore

$G \cap F = \{0\}$.

For non-singularity of $g|_G$, suppose that $z \in G$ such that $g(z,x) = 0$ for every $x \in G$. This implies that $z \perp G$ and $z \in F$.

Consequently $z \in G \cap F = \{0\}$, i.e. $z = 0$ and this shows the non-singularity of $g|_G$. By construction of a base in G it is easy to prove that $g(e_i, e_j) = \delta_{ij}$ for every $i,j = 1,\ldots,k$. ([2]).

Finally, it remains to prove that for every $z \in E$, $z = x + y$ where $x = \lambda^i e_i \in G$, $i = 1,\ldots,k$, $y \in F$. This means that we must determine $\lambda^1,\ldots,\lambda^k \in \mathbb{R}$ such that $y = z - \lambda^i e_i \in F$ which is equivalent to $(z - \lambda^i e_i) \perp e_j$, $j = 1,\ldots,k$.

Consequently $g(z - \lambda^i e_i, e_j) = 0$ and $g(z, e_j) = \lambda^j$.

Therefore, for every $z \in E$, $z = \sum_{i=1}^{k} g(z, e_j).e_j + y$ such that $\sum_{i=1}^{k} g(z, e_j).e_j \in G$, $y \in F$ which means that $E \subset G \oplus F$ and this completes the proof of the theorem.

## REFERENCES

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2- Taylor, A. E., and Lay D. C., Introduction To Functional Analysis, John Wilely & Sons, Inc 1980.

## البحث الأول : "بحث منفرد"

**إسم البحث:**

On Hyperplanes in Banach Spaces

**عن المستويات الزائدية فى فراغات بناخ**

**المجلة :**

**مجلة كلية العلوم جامعة المنوفية – مصر**

Sci. J. Fac. Sci. , Menoufia University, Shebin El-Koom,
EGYPT, VOL. VIII, No. 58, 1994

**نبذة عن البحث:-**

يتناول البحث بالدراسة الفراغ الجزئى F من فراغ (بناخ ) E لانهائى البعد ولقد تم فى هذا البحث التوصل إلى نتيجة مضمونها أنه إذا وجدت صيغة (Form ) g معرفة على E ولها خاصية قوة عدم الإنفراد ( Strongly non - Singular ) بحيث كانت الصيغة g المقيدة على الفراغ الجزئى F ضعيفة عدم الإنفراد ( Weakly non- Singular) فإنه يوجد فراغ جزئى G عمودى على F بحيث تكون الصيغة g المقيدة على هذا الفراغ الجزئى G ضعيفة عدم الإنفراد . و لقد إستخدمت هذة النتيجة فى تعميم العديد من الخواص الهندسية للفراغات متعددة الطيات الجزئية لانهائية البعد F والتى لها الخاصية( Codim F= k < )