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SOLUTIONS OF HEAT CONDUCTION PROBLEMS IN A SEMI-INFINITE MACRO-PERIODIC LAYERED MEDIA USING THE EFFECTIVE MODULUS MODEL

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ABSTRACT

The problem of heat conduction in a semi-infinite macro-periodic layered media is treated by applying the effective modulus model. This model describes the micro-morphic effects or the effects due to the micro-periodic structure of the rigid body on the transient heat conduction. In such model the problem is described by two coupled partial differential equations. The Laplace transformation with respect to the time is used to solve these equations. Inversion of the resulting expressions is carried out using the convolution theorem. Two different problems have been solved. The first problem is of the first kind of the boundary conditions while the second problem is of the second kind. Numerical analysis of the obtained solutions is also presented for two different kinds of two alternating layers.

INTRODUCTION

The study of heat conduction through multilayered media seized a great attention during the last two decades. This is due to the numerous applications in engineering fields such as design of the wall of industrial

furnaces, rockbeds, chemical reactors, nuclear reactor and constructions.

A variety of purely numerical methods have handled successfully the heat flow calculation in multilayered mediums. Nevertheless, analytical approaches remain of interest to designers since they give more synthetic insight into the influence of each parameter. The analytical methods are generally based on separation of variables in the heat equation, Green's functions and integral transformation (Fourier and Laplace transformations) [1, 2].

Many papers have been presented to study heat conduction problems in periodic laminated composite. Furmanski [3] studied the heat conduction in a two-component medium for both regular (periodic) or irregular (chaotic) inner structure of the medium. The energy balance equation which describes the heat transfer processes in two components was derived using the ensemble-averaging technique. Some examples of application of the theory are presented in calculation of the interaction coefficient for different kind of composite materials (periodic laminated composite, composite reinforced with unidirectionally aligned fibbers and composite with dispersion of spherical inclusions).

Auriault et al. [4] presented a macroscopic modeling of heat transfer in periodic composites in the presence of interfacial thermal resistance .The method of double-scale asymptotic developments is used to determine the interfacial thermal resistance. Five characteristic cases are considered related to different relative values of the barrier resistance to the resistance of the components. The first three models are one-temperature field models whereas the last two are two-temperature field models. To illustrate the five corresponding macroscopic models, a layered medium is investigated which permits analytical results.

Wozniak et. al. [5, 6] presented a new approach to modelling of nonstationary heat conduction problems in micro-periodic composites. The refined averaged model of a rigid heat conductor with micro-periodic structure is formulated. This model describes the effect of a microstructure length dimension in the time-dependent heat transfer problems.

Ignaczak et. al. [7] proposed a formulation for one-dimensional boundary value problem for a periodically layered plate using the refined averaged heat conduction theory. The uniqueness theorem for the problem is proved under the sufficient condition upon which the second law of thermodynamics is satisfied for the smeared layer problem and using a global thermal energy conservation law associated with the problem. Also, solutions of two particular initial boundary value problems are presented. One of the two particular solutions represents a temperature field in the layered semispace due to sudden heating of the boundary plane, while the other stands for a temperature field in the semispace produced by a laser surface heating.

Matysiak et. al. [8] considered the problem of transient heat conduction in a periodically stratified medium consisting of a large number of alternating concentric cylinders of two homogeneous isotropic rigid materials and in a rotationally periodic cylinder consisting of a large number of circular homogeneous isotropic rigid sectors. The equation of the homogenized models with microlocal parameters are derived taking into account certain microlocal effects connected with the microperiodic structure of the considered composites. Some problems of temperature distribution in composite cylinders are considered as an application to the presented models.

The problem of heat condition in a half-space medium consisting of a periodic two alternating layers is considered by El-Zebidy [9]. The problem is treated in frame of the refined averaged theory. The solution for the temperature distribution is obtained by using Laplace transformation technique. Inversion of the resulting expression is carried out using the residues theorem. Numerical analysis of obtained solutions of the problem is also presented for born-epoxy layers.

In this paper, the problem of heat conduction in a semi-infinite periodically multilayered medium is solved in frame of effective modulus theory. Section two contains the basic equations of the effective modulus theory. In section three, we present a solution of the problem of heat conduction in a semi-infinite periodically multilayered composite subjected to a boundary condition of the first kind. In section four the same problem which presented in section three is solved but the boundary condition is of the second kind. In section five, a numerical analysis and conclusions of the solutions of the two problems are presented.

THE EFFECTIVE MODULUS MODEL

The effective modulus model theory [10] takes into account certain micromorfic effects resulting from the fine periodic structure of the body. If we consider a rigid body occupies a regular region Ω in the Euclidean 3-space referred to Cartesian coordinates system. The Cartesian coordinates of points of Ω will be denoted by $\mathrm{x} = (x^i)$, $i = 1,2,3$. We also introduce in Ω a system of curvilinear coordinates $\mathrm{X} = (X^\alpha)$, $\alpha = 1,2,3$. Setting $\mathrm{x} = \mathrm{x}(\mathrm{X})$, $\mathrm{X} \in \Omega_R$ where Ω_R is

a regular region in R^3. Functions $x(.) = (x_i(.))$ are assumed to be single valued and continuously differentiable except possibly at some points, lines and surfaces.

Let $\theta(X,t) = T(X,t) + T_0$, $t \in R$ be the absolute temperature field defined in Ω, where T_0 is known constant reference temperature. Also let $A(X) = (A_i^\alpha(X))$ be a matrix inverse to $\nabla x(X) = (x^i_{,\alpha}(X))$. The heat flux relation at $x = x(X)$ and at a time instant t will be assumed in the form

$$h^i(X,t) = K^{ij}(X,T(X,t))A_j^\alpha(X)T_{,\alpha}(X,t),$$

where $K^{ij}(.) = K^{ji}(.)$ are the known components of the thermal conductivity tensor. The curvilinear coordinates $X = (X^\alpha) \in \Omega_R$ will be assumed that they are related to a free energy function $\varphi(X,T(X,t))$ and the heat generation $g(X,T(X,t),t)$.
Define

$$K^{\alpha\beta}(X,T) = A_i^\alpha(X)A_j^\beta(X)K^{ij}(X,T),$$

The energy conservation principle has the form:

$$h^\alpha\big|_\alpha(X,t) + g(X,T(X,t),t) = T(X,t)\partial_t\mu(X,t), \tag{2.1}$$

where $\mu(X,t)$ is the entropy, the vertical line stands for the covariant differentiation in the metric tensor

$$G(X) \equiv (G_{\alpha\beta}(X)) = (x^i_{,\alpha}(X)x_{i,\beta}(X)), \quad X \in \Omega_R,$$

and where

$$h^\alpha(X,t) = K^{\alpha\beta}(X,T(X,t))T_{,\beta}(X,t),$$

$$\mu(X,t) = \frac{-\partial\varphi(X,T(X,t))}{\partial T(X,t)} \tag{2.2}$$

Esq. (2.1), (2.2) are assumed to hold almost every where in Ω (i.e. for a.e. $X \in \Omega_R$ for which $K^{\alpha\beta}(X,t)$ are defined) and for every $t \in R$.

Let (ξ^1,ξ^2,ξ^3) be a triplet of positive numbers, and define $\xi_\alpha \equiv (\delta^1_\alpha\xi^1,\delta^2_\alpha\xi^2,\delta^3_\alpha\xi^3)$, $\alpha=1,2,3$. In order to express the concept "micro-periodic" medium, the following conditions will be assumed

1- For every $\mathbf{Z}\in\Omega_R$, such that $\mathbf{Z}+\xi_\alpha\in\Omega_R$, we have

$$\rho(\mathbf{Z}) \quad = \rho(\mathbf{Z}+\xi_\alpha),$$

$$\varphi(\mathbf{Z},T) \quad = \varphi(\mathbf{Z}+\xi_\alpha,T),$$

$$K^{ij}(\mathbf{Z},T) = K^{ij}(\mathbf{Z}+\xi_\alpha,T),$$

where $\rho(X), X\in\Omega_R$ is the mass density.

2- The maximum distance $d(\mathrm{x}(\mathrm{X}),\mathrm{x}(\mathrm{X}+\xi_\alpha)), \mathrm{X}\in\overline{\Omega}_R$, is much smaller than l_R

$$\max\ d\big(\mathrm{x}(\mathrm{X}),\mathrm{x}(\mathrm{X}+\xi_\alpha)\big) << l_R,$$

where l_R is the smallest characteristic length dimension of region Ω_R in R^3.

3- For every $\Delta\mathrm{X}=(\Delta X^1,\Delta X^2,\Delta X^3)$, such that $|\Delta X^\alpha|<\xi^\alpha,\alpha=1,2,3$, then

$$\mathrm{x}(\mathrm{X}+\Delta\mathrm{X}) \cong \mathrm{x}(\mathrm{X})+\nabla\mathrm{x}(\mathrm{X})\Delta\mathrm{X},$$

holds in Ω_R.

Under aforementioned conditions the body under consideration will be called a microperiodic composite. It has to be emphasized that a term "periodic" is related to curvilinear coordinates $\mathrm{X}=(X^\alpha)$.

The effective modulus model is based on the assumption that the temperature field has the form

$$T(\mathrm{X},t)=T^0(\mathrm{X},t)+\eta^a(\mathrm{X})T^a(\mathrm{X},t),\quad \mathrm{X}\in\Omega_R,\ t\in R, \tag{2.3}$$

where the functions $\eta^a(.)$, which called shape functions are known continuous and differentiable almost everywhere functions, such that

$$\eta^a(\mathrm{X})=\eta^a(\mathrm{X}+\xi_\alpha),$$

$$\int_0^{\xi_\alpha} \eta^a(\mathbf{X})dX^\alpha = 0, \quad \alpha = 1,2,3. \tag{2.4}$$

The function $T^0(.,t)$ is called macro-temperature and the functions $T^a(.,t)$ are called the micromorphic parameters.

Let $\mathbf{X}_0 = (X_0^\alpha)$ be an arbitrary but fixed point of, such that

$$C(\mathbf{X}_0) := \{Z \in \Omega_R \mid |Z^\alpha - X^\alpha| < 0.5\xi^\alpha, \alpha = 1,2,3\}$$

is a subset of Ω_R. The average operator is introduced in the form

$$\langle\psi(.)\rangle = \frac{1}{V}\int_{C(\mathbf{X}_0)} \psi(\mathbf{Z})dV(\mathbf{Z}), \tag{2.5}$$

where $V \equiv \xi^1\xi^2\xi^3$ and $dV(\mathbf{Z}) = dZ^1dZ^2dZ^3$.

In this model the The conduction equations after linearization take the form [10]

$$\tilde{h}^\alpha|_\alpha(\mathbf{X},t) + \tilde{g}(\mathbf{X},t) = \langle\rho c\rangle\partial_t T^0(\mathbf{X},t),$$
$$g^A(\mathbf{X},t) = 0, \tag{2.6}$$

where

$$\tilde{h}^\alpha(\mathbf{X},t) = \tilde{K}^{\alpha\beta}(\mathbf{X})T^0_{,\beta}(\mathbf{X},t) + K^{A\alpha}(\mathbf{X},t)T^A(\mathbf{X},t),$$
$$g^A(\mathbf{X},t) = K^{A\alpha}(\mathbf{X})T^0_{,\alpha}(\mathbf{X},t) + L^{AB}(\mathbf{X},t)T^B(\mathbf{X},t), \tag{2.7}$$

and

$$\tilde{K}^{\alpha\beta}(\mathbf{X},T) = A_i^\alpha(\mathbf{X})A_j^\beta(\mathbf{X})\langle k^{ij}(\mathbf{Z})\rangle,$$
$$K^{A\alpha}(\mathbf{X}) = A_i^\alpha(\mathbf{X})A_j^\beta(\mathbf{X})\langle k^{ij}(\mathbf{Z},T)\eta^A_{,\beta}(\mathbf{Z})\rangle, \tag{2.8}$$
$$L^{AB}(\mathbf{X}) = A_i^\alpha(\mathbf{X})A_j^\beta(\mathbf{X})\langle k^{ij}(\mathbf{Z})\eta^A_{,\alpha}(\mathbf{Z})\eta^B_{,\beta}(\mathbf{Z})\rangle.$$

In the case of an isotropy there is $K_0^{ij}(\mathbf{X}) = K(\mathbf{X})\delta^{ij}, \mathbf{X} \in \Omega_R$ where $k(\mathbf{X})$ is the value of the heat conductivity coefficient. If both systems of coordinates $(X^\alpha), (x^i)$ coincide, then $A_i^\alpha(x) = \delta_i^\alpha, G^{\alpha\beta}(X) = \delta^{\alpha\beta}$, and

coefficients $\widetilde{K}^{\alpha\beta}$, $K^{A\alpha}$, L^{AB} in Eqs. (2.7) and (2.8) are independent of $\mathbf{X}$. In this case the governing equations take the form

$$\widetilde{h}^{i}_{,i}(\mathbf{x},t)+\widetilde{g}(\mathbf{x},t)=\langle\rho c\rangle\partial_t T^0(\mathbf{X},t),$$

$$g^A(\mathbf{x},t)=0, \qquad (2.9)$$

where

$$\widetilde{h}^{i}(\mathbf{x},t)=\langle k\rangle T^0_{,i}(\mathbf{x},t)+\langle k\eta^A_{,i}\rangle T^A(\mathbf{x},t),$$

$$g^A(\mathbf{x},t)=\langle k\eta^A_{,i}\rangle T^0_{,i}(\mathbf{X},t)+\langle k\eta^A_{,i}\eta^B_{,i}\rangle T^B(\mathbf{x},t), \qquad (2.10)$$

PERIODICALLY HEATED SURFACE

Consider one-dimensional problem of composite semi-space with initial temperature equal to zero. The surface temperature is changed to $T_0\cos\omega t$ where T_0,ω are constant. The medium consists of alternating layers of two materials having thermal conductivities k_1,k_2, densities ρ_1 and ρ_2 and specific heat c_1 and c_2 respectively. The thickness of the first layer is l_1 and the thickness of the second layer is l_2, where $l\ +l_2=l$ is the thickness of the period.

Solution of the problem

From Eqs. (2.9), (2.10) we have the governing equations in the form

$$\langle k(x)\rangle T^0_{,xx}(x,t)+\langle k(x)\eta_{,x}(x)\rangle T^1_{,x}(x,t)-\langle\rho(x)c(x)\rangle\dot{T}^0(x,t)=0,$$

$$x,t>0 \qquad (3.1)$$

$$\langle k(x)\eta_{,x}(x)\rangle T^0_{,x}(x,t)+\langle k(x)\eta_{,x}(x)\eta_{,x}(x)\rangle T^1(x,t)=0,$$

with the initial and boundary conditions

$$T^0(x,t)=T^1(x,t)=0 \qquad \text{at } t=0,$$

$$T^0(x,t)=T_0\cos\omega t \qquad \text{at } x=0, \qquad (3.2)$$

$$T^0(x,t)\to 0 \qquad \text{as } x\to\infty,$$

where $\eta(x)$, $\langle k(x)\rangle$, $\langle \rho(x)c(x)\rangle$, $\langle k(x)\eta_{,x}(x)\rangle$, and $\langle k(x)\eta_{,x}(x)\eta_{,x}(x)\rangle$ are defined as follows [11]

$$\eta(x)=\begin{cases} -\dfrac{l}{4l_1}x+\dfrac{l}{8} & 0\le x\le l_1, \\ \dfrac{l}{4l_2}x-\dfrac{ll_1}{4l_2}-\dfrac{l}{8} & l_1\le x\le l, \end{cases}$$

$$\langle k(x)\rangle=\frac{1}{l}\int_0^l k(x)dx,$$

$$\langle \rho(x)c(x)\rangle=\frac{1}{l}\int_0^l \rho(x)c(x)dx, \tag{3.3}$$

$$\langle k(x)\eta_{,x}(x)\rangle=\frac{1}{l}\int_0^l k(x)\eta_{,x}(x)dx,$$

$$\langle k(x)\eta_{,x}(x)\eta_{,x}(x)\rangle=\frac{1}{l}\int_0^l k(x)\eta_{,x}(x)\eta_{,x}(x)dx,$$

Introducing the following non-dimensional parameters:

$$\xi=\frac{x}{l}, \qquad \tau=\frac{\langle k(x)\rangle t}{\langle \rho(x)c(x)\rangle l^2},$$

$$\vartheta^0(\xi,\tau)=\frac{T^0(x,t)}{T_0}, \qquad \vartheta^1(\xi,\tau)=\frac{T^1(x,t)}{T_0},$$

then, the Eqs. (3.1), (3.2) take the form

$$\vartheta^0_{,\xi\xi}(\xi,\tau)+\frac{\langle k(\xi)\eta_{,\xi}(\xi)\rangle}{\langle k(\xi)\rangle}\vartheta^1_{,\xi}(\xi,\tau)-\dot{\vartheta}^0(\xi,\tau)=0,$$

$$\vartheta^1(\xi,\tau)+\frac{\langle k(\xi)\eta_{,\xi}(\xi)\rangle}{\langle k(\xi)\eta_{,\xi}(\xi)\eta_{,\xi}(\xi)\rangle}\vartheta^0(\xi,\tau)=0, \tag{3.4}$$

and the initial and boundary conditions become

$$\begin{aligned} &\vartheta^0(\xi,\tau)=\vartheta^1(\xi,\tau)=0 && \text{at } \tau=0, \\ &\vartheta^0(\xi,\tau)=\cos\omega\tau && \text{at } \xi=0, \\ &\vartheta^0(\xi,\tau)\to 0 && \text{as } \xi\to\infty. \end{aligned} \tag{3.5}$$

where

$$\omega_1=\frac{\omega l^2\langle\rho(\xi)c(\xi)\rangle}{\langle k(\xi)\rangle}$$

Taking the Laplace transformation to Eqs. (3.4) and (3.5), we have

$$\begin{aligned} &\bar{\vartheta}^0_{,\xi\xi}(\xi,p)+\alpha_1\bar{\vartheta}^1_{,\xi}(\xi,p)-p\bar{\vartheta}^0(\xi,p)=0 \\ &\bar{\vartheta}^1(\xi,\tau)+\beta_1\bar{\vartheta}^0_{,\xi}(\xi,p)=0, \end{aligned} \tag{3.6}$$

and the initial and boundary conditions become

$$\begin{aligned} &\bar{\vartheta}^0(\xi,p)=\bar{\vartheta}^1(\xi,p)=0 && \text{at } \tau=0, \\ &\bar{\vartheta}^0(\xi,p)=\frac{p}{p^2+\omega_1^2} && \text{at } \xi=0, \\ &\bar{\vartheta}^0(\xi,p)=0 && \text{as } \xi\to\infty, \end{aligned} \tag{3.7}$$

where

$$\alpha_1=\frac{\langle k(\xi)\eta_{,\xi}(\xi)\rangle}{\langle k(\xi)\rangle}, \qquad \beta_1=\frac{\langle k(\xi)\eta_{,\xi}(\xi)\rangle}{\langle k(\xi)\eta_{,\xi}(\xi)\eta_{,\xi}(\xi)\rangle} \tag{3.8}$$

From Eq $(3.6)_2$ we have

$$\bar{\vartheta}^1(\xi,p)=-\beta_1\bar{\vartheta}^0_{,\xi}(\xi,p) \tag{3.9}$$

Eliminating $\bar{\vartheta}^1(\xi,p)$ from Eqs. (3.6), we have

$$\bar{\vartheta}^0_{,\xi\xi}(\xi,p)=\frac{p}{(1-\alpha_1\beta_1)}\bar{\vartheta}^0(\xi,p) \tag{3.10}$$

The solution of eq. (3.10) which satisfies the boundary conditions (3.7) is

$$\overline{\vartheta}^0(\xi,p)=\frac{p}{p^2+\omega_1{}^2}\exp(-\xi\sqrt{\frac{p}{1-\alpha_1\beta_1}}) \tag{3.11}$$

In order to inverte the above function to the original variable τ we use the convolution theorem and tables of the inverse Laplace transformation [12], then we get

$$\vartheta^0(\xi,\tau)=\frac{\xi}{2\sqrt{\pi(1-\alpha_1\beta_1)}}\int_0^{\tau}\cos[\omega_1(\tau-u)]\ (u)^{\frac{-3}{2}}\exp(-\frac{\xi^2}{4u((1-\alpha_1\beta_1)})du\,. \tag{3.12}$$

From Eqs. (3.9) and (3.12) we get

$$\overline{\vartheta}^1(\xi,p)=\frac{\beta_1\,p\sqrt{p}}{(p^2+\omega_1^2)\sqrt{(1-\alpha_1\beta_1)}}\exp(-\xi\sqrt{\frac{p}{1-\alpha_1\beta_1}}) \tag{3.13}$$

In the same manner we can obtain $\vartheta^1(\xi,\tau)$ in the form

$$\vartheta^1(\xi,\tau)=\frac{\beta_1}{4\sqrt{\pi(1-\alpha_1\beta_1)}}\int_0^{\tau}(\frac{\xi^2}{(1-\alpha_1\beta_1)}-2u)(u)^{\frac{-5}{2}}\times$$
$$\exp(\frac{-\xi^2}{4u(1-\alpha_1\beta_1)})\cos[\omega_1(\tau-u)]du\,. \tag{3.14}$$

Differentiating Eq. (3.12) with respect to ξ, we obtain the gradient of the macro-temperature in the form

$$\vartheta^0_{,\xi}(\xi,\tau)=\frac{-1}{4\sqrt{\pi(1-\alpha_1\beta_1)}}\int_0^{\tau}(\frac{\xi^2}{(1-\alpha_1\beta_1)}-2u)(u)^{\frac{-5}{2}}\times$$
$$\exp(\frac{-\xi^2}{4u(1-\alpha_1\beta_1)})\cos[\omega_1(\tau-u)]du \tag{3.15}$$

Notice that one can obtain the solution for the homogeneous medium from the above solution. For the homogeneous medium $\alpha_1=\beta_1=0$, from Eq. (3.14) we find that $\vartheta^1(\xi,\tau)=0$, and the temperature in homogeneous medium $\vartheta_h(\xi,\tau)$ from Eq. (3.12) takes the form

$$\vartheta_h\ (\xi,\tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^{\tau} \cos[\omega_1(\tau-u)]\ (u)^{\frac{-3}{2}} \exp(-\frac{\xi^2}{4u})du, \qquad (3.16)$$

which is the temperature in a semi-infinite homogeneous medium has the same initial and boundary conditions of this problem.

TEMPERATURE IN A LAYERED SEMISPACE SUBJECT TO STEADY PERIODIC SURFACE HEAT FLUX

Let us consider the same problem treated in section 3 but in this problem the initial temperature of the body is T_0 where T_0 is a constant and the surface temperature is subjected to a heat flux given by $-\langle k\rangle \frac{\partial T(0,t)}{\partial x} = a\, Q_0 \cos\omega t$, where a is a controlling factor for steady periodic oscillations and Q_0 is a constant surface heat flux.

Solution of the problem

As in the Section 3, we have the governing equations in the form

$$\begin{aligned} &\bar{\vartheta}^0_{,\xi\xi}(\xi,p) + \alpha_1 \bar{\vartheta}^1_{,\xi}(\xi,p) - p\bar{\vartheta}^0(\xi,p) + 1 = 0, \\ &\bar{\vartheta}^1(\xi,p) + \beta_1 \bar{\vartheta}^0_{,\xi}(\xi,\tau) = 0, \end{aligned} \qquad (4.1)$$

and the initial and boundary conditions become

$$\begin{aligned} &\bar{\vartheta}^0(\xi,p) = \frac{1}{p} && \text{at } \tau = 0, \\ &-\langle k\rangle \bar{\vartheta}^0_{,\xi}(\xi,p) = aQ_0 \frac{p}{p^2+\omega_1^2} && \text{at } \xi = 0, \\ &\bar{\vartheta}^0(\xi,p) \text{ is finite as} && \xi \to \infty, \end{aligned} \qquad (4.2)$$

where α_1, β_1 have the same definition as in section 3 and

$$\omega_1 = \frac{\omega l^2 \langle \rho(\xi) c(\xi)\rangle}{\langle k(\xi)\rangle}$$

From Eq (3.15)$_2$ we have

$$\bar{\vartheta}^1(\xi,p) = -\beta_1 \bar{\vartheta}^0_{,\xi}(\xi,p) \tag{4.3}$$

Eliminating $\bar{\vartheta}^1(\xi,p)$ from Eqs. (4.1), yields

$$\bar{\vartheta}^0_{,\xi\xi}(\xi,p) = \frac{p}{(1-\alpha_1\beta_1)}\bar{\vartheta}^0(\xi,p) - \frac{1}{(1-\alpha_1\beta_1)} \tag{4.4}$$

The solution of Eq. (4.4) which satisfies the boundary conditions (4.2) is

$$\bar{\vartheta}^0(\xi,p) = H\sqrt{1-\alpha_1\beta_1}\,\frac{\sqrt{p}}{p^2+\omega_1^2}\exp(-\xi\sqrt{\frac{p}{1-\alpha_1\beta_1}} + \frac{1}{p}, \tag{4.5}$$

where $H = \dfrac{aQ_0}{\langle k(\xi)\rangle}$.

In order to inverte the above function to the original variable τ we use the convolution theorem and tables of the inverse Laplace transformation, then we get

$$\vartheta^0(\xi,\tau) = \frac{H\sqrt{1-\alpha_1\beta_1}}{2\omega_1\sqrt{\pi}}\int_0^\tau (\frac{\xi^2}{(1-\alpha_1\beta_1)} - 2u)(u)^{-\frac{5}{2}} \times$$
$$\exp(\frac{-\xi^2}{4u(1-\alpha_1\beta_1)})\sin[\omega_1(\tau-u)]du + 1 \tag{4.6}$$

From Eqs. (4.3), (4.6) we get

$$\bar{\vartheta}^1(\xi,p) = \frac{\beta_1 H\, p}{p^2+\omega_1^2}\exp\left(-\xi\sqrt{\frac{p}{1-\alpha_1\beta_1}}\right) \tag{4.7}$$

In the same manner we can obtain $\vartheta^1(\xi,\tau)$ in the form

$$\vartheta^1(\xi,\tau) = \frac{\beta_1 H\,\xi}{2\sqrt{\pi(1-\alpha_1\beta_1)}}\int_0^\tau (u)^{\frac{-3}{2}}\exp\left(\frac{-\xi^2}{4u(1-\alpha_1\beta_1)}\right)\cos[\omega_1(\tau-u)]du \tag{4.8}$$

Differentiating Eq. (4.9) with respect to ξ, we obtain the gradient of the macro-temperature in the form

$$\vartheta^{0}_{,\xi}(\xi,\tau)=\frac{H\xi}{4\omega_1\sqrt{\pi(1-\alpha_1\beta_1)}}\int_0^{\tau}\left(3-\frac{\xi^2}{2u(1-\alpha_1\beta_1)}\right)u^{\frac{-5}{2}}\times \exp\left(\frac{-\xi^2}{4u(1-\alpha_1\beta_1)}\right)\sin[\omega_1(\tau-u)]du\,. \tag{4.9}$$

Notice that one can obtain the solution for the homogeneous medium from the above solution. For the homogeneous medium $\alpha_1=\beta_1=0$, from Eq. (4.8) we find that $\vartheta^1(\xi,\tau)=0$, and the temperature in homogeneous medium $\vartheta_h(\xi,\tau)$ from Eq. (4.6) takes the form

$$\vartheta_h(\xi,\tau)=\frac{H}{2\omega_1\sqrt{\pi}}\int_0^{\tau}(\xi^2-2u)(u)^{-\frac{5}{2}}\exp(\frac{-\xi^2}{4u})\sin[\omega_1(\tau-u)]du\,, \tag{4.10}$$

which is the temperature in a semi-infinite homogeneous medium has the same initial and boundary conditions of this problem.

NUMERICAL RESULTS AND CONCLUSIONS

In this section we present a numerical analysis for the analytical solutions obtained in the pervious two sections. It is assumed that the medium consists of two alternating layers. The thickness of the second layer is three times the first layer. Then the shape function given by Eqs. (3.3)$_1$ will take the form

$$\eta(x)=\begin{cases}-x+\dfrac{l}{8} & 0\le x\le l/4,\\[2ex] \dfrac{x}{3}-\dfrac{5l}{24} & l/4\le x\le l,\end{cases} \tag{5.1}$$

and so the non-dimensional shape function takes the form

$$\eta(\xi)=\begin{cases}-\xi+\dfrac{1}{8} & 0\leq\xi\leq 1/4,\\ \dfrac{\xi}{3}-\dfrac{5}{24} & 1/4\leq\xi\leq 1.\end{cases} \qquad (5.2)$$

The numerical results are presented for two types of materials. These types are

1. **Boron-Epoxy**
2. **Boron-Aluminum**

The physical properties of these materials are presented in table 5.1 and the numerical values of the parameters α_1,β_1 are presented in table 5.2.

Table 5.1 the physical properties of the materials used in calculations.

Substance	Conductivity k cal/sec cm °C	Density ρ gm/cm^3	Specific heat c cal/gm °C
Boron	7.648×10^{-3}	2.32	0.2451
Epoxy	8.365×10^{-4}	1.14	0.4498

Table 5.2 The numerical values of the parameters used in the effective modulus model.

	α_1	β_1
Boron-Epoxy	-0.670588235	-0.859296482
Boron- Aluminum	0.326398485	2.820238892

In Figs. 1 through Figs. 6, the curves labeled with the numbers 1, 2, 3 are referred to the values calculated at the first, second and third interfaces of the layers respectively. Figures 1, 2 and 3 are related to the problem of

periodically heated surface while figures 4, 5and 6 are related to the problem of steady periodic heat surface heat flux.

Figs. 1 show the non-dimensional macro-temperature $\vartheta^0(\xi,\tau)$ given by Eq. (3.12) versus the non-dimensional time τ. The macro-temperature is calculated at the first three interfaces of the layers. The results obtained for the value $\omega_1 = 0.1$. Figs. 2 illustrate the dependence of the total temperature $\vartheta(\xi,\tau)$ at the first three interfaces on the non-dimensional time τ. Figs. 3 present the variation of the oscillation of temperature gradient $\Delta\vartheta_{,\xi}(\xi,\tau)$ between adjacent layers with respect to the non-dimensional time τ. Also, the calculated values are presented at the first three interfaces of the layers.

In Figs. 4 we present the variation of the non-dimensional macro-temperature which is given by Eq. (4.6) with respect to the non-dimensional time τ. The macro-temperature is calculated at the first three interfaces of the layers. The results obtained for the value $\omega_1 = 0.1$ and $Q_0 = 4.7769179 \times 10^{-3}$ Cal/Sec.cm^2. In Figs. 5 the non-dimensional total temperature $\vartheta(\xi,\tau)$ is plotted as a function of the non-dimensional time τ. The results are given for the values at the first three interfaces of the layers. Figs. 6 present the variation of the oscillation of temperature gradient $\Delta\vartheta_{,\xi}(\xi,\tau)$ between adjacent layers with respect to the non-dimensional time τ. The calculated values are presented at the first three interfaces of the layers.

The main feature of the effective modulus model is that it describes the micro-morphic effects in a temperature distribution due to micro-periodic structure of the body. From the numerical results which are presented in figures 1-6 we can coclude that

- For the laminate consists of materials of low thermal conductivity, the values of the total temperature is very close to

the values of the macro-temperature $\vartheta^0(\xi,\tau)$ and then the values of the $\vartheta^1(\xi,\tau)$ are very small and can be neglected. But this is not true for the laminate, which contains at least one material of high thermal conductivity, where we note that the values of $\vartheta^1(\xi,\tau)$ are not small and can not be neglected.

- **For the medium made of laminates consist of two materials, both have low thermal conductivity, the values of the total temperature $\vartheta(\xi,\tau)$ are higher compared with the total temperature of the medium if one material is exchanged by a material of high thermal conductivity.**

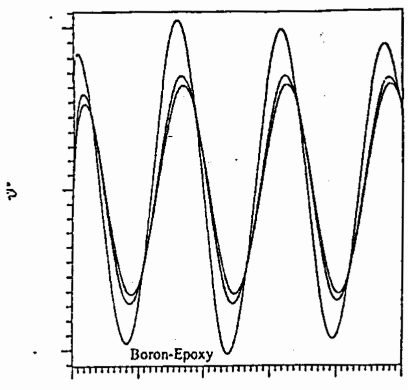

Fig. .a The variation of ϑ^* with the time τ at the first three interfaces.

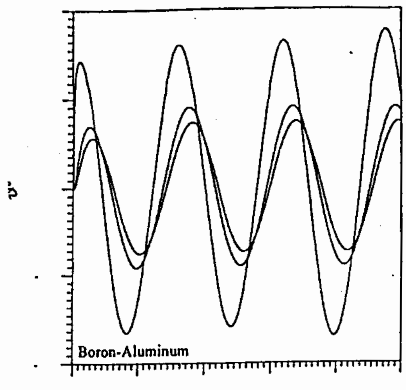

Fig. .b The variation of ϑ^* with the time τ at the first three interfaces.

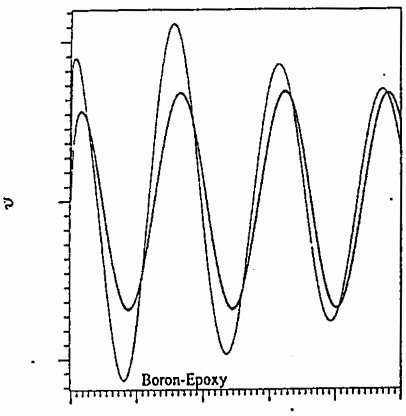

Fig. .a The variation of ϑ with the time τ at the first three interfaces.

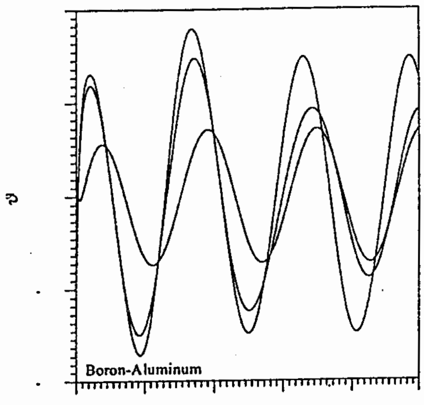

Fig. .b The variation of ϑ with the time τ at the first three interfaces.

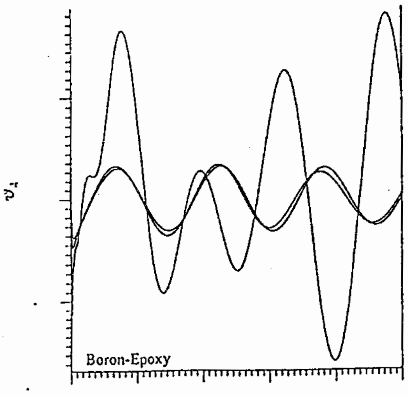

Fig. .a The variation of ϑ_4 with the time τ at the first three interfaces.

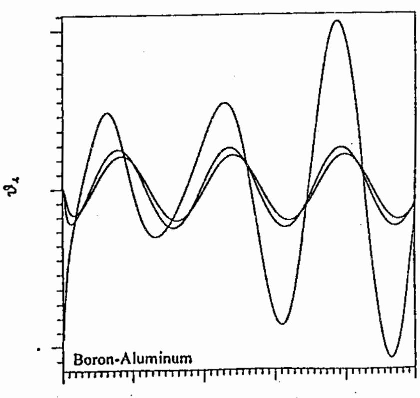

Fig. .b The variation of ϑ_4 with the time τ at the first three interfaces.

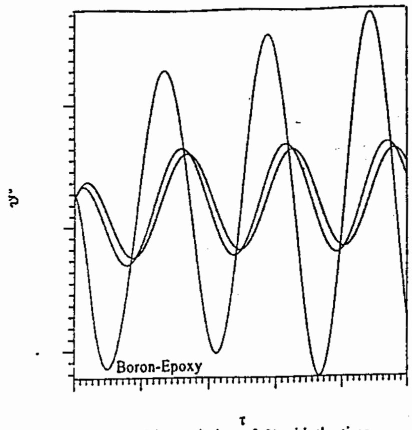

Fig. .a The variation of ϑ^* with the time τ at the first three interfaces.

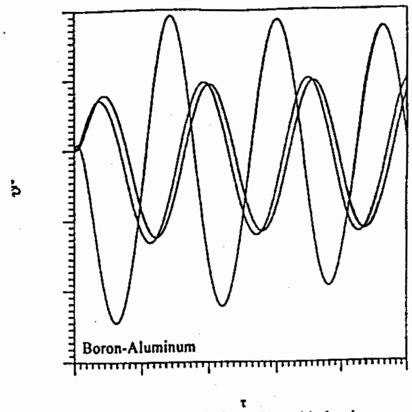

Fig. .b The variation of ϑ^* with the time τ at the first three interfaces.

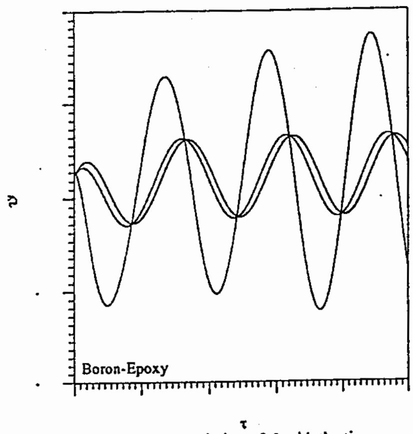

Fig. .a The variation of ϑ with the time τ at the first three interfaces.

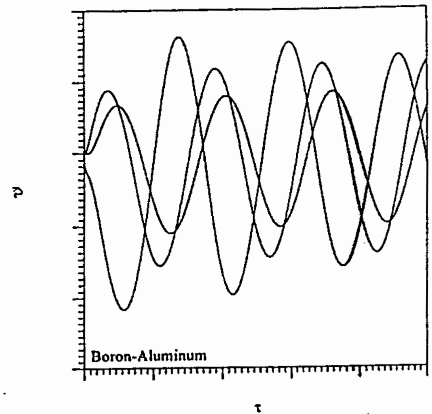

Fig. .b The variation of ϑ with the time τ at the first three interfaces.

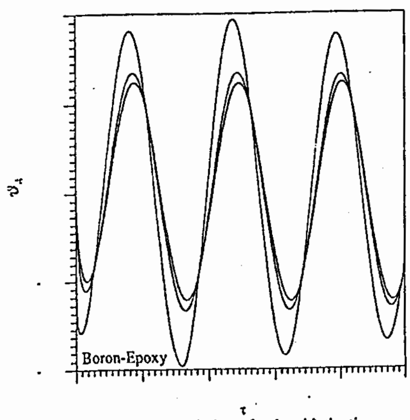

Fig. .a The variation of ϑ_A with the time τ at the first three interfaces.

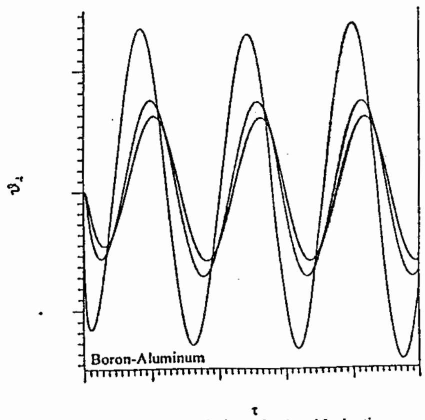

Fig. .b The variation of ϑ_A with the time τ at the first three interfaces.

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في هذا البحث تم دراسة التوصيل الحراري في وسط متعدد الطبقات الرقيقة دوريا في إطار (نموذج المعامل المؤثر) والذي يعتمد بشكل أساسي على دراسة التأثير الناتج عن طبيعة تكوين الجسم من طبقات رقيقة من مواد مختلفة في الخواص الفيزيائية و الحرارية على انتقال الحرارة خلال الجسم. و نظرا لتذبذب الخواص الحرارية للمواد المؤلفة (مثل معامل التوصيل الحراري) فان معادلات الحرارة الكلاسيكية لجسم مصنوع من مواد مؤلفة لا يمكن استخدامها بصوره مباشرة لوصف عمليه انتقال الحرارة في الجسم. وبالتالي فان الهدف الأساسي من تطبيق (نموذج المعامل المؤثر) هو استنتاج مجموعه جديده من المعادلات من معادلات الحرارة الكلاسيكية تتحول فيها جميع الدوال التي تعبر عن الخواص الفيزيائية و الحرارية للجسم إلى القيمة المتوسطة لهذه الدوال و التي لا تعتمد على الإحداثيات. و بفرض أن انتقال الحرارة يحدث بشكل تام بين طبقات الجسم تم استنتاج المعادلات الأساسية الخاصة بنموذج (المعامل المؤثر) من معادلات الحرارة الأولية وذلك باستخدام طرق التحليل الغير قياسى. في إطار هذه المعادلات تم صياغة مسائل التوصيل الحراري في وسط متعدد الطبقات الرقيقة دوريا . باستخدام الطرق التحليلية و التي تعتمد على استخدام تحويلات لابلاس العكسية و نظرية الاندماج تم إيجاد صيغ لحساب درجة الحرارة و الفيض الحراري لمسائل التوصيل الحراري وذلك بوضع مجموعة من الشروط الحدية و الابتدائية.

في نهاية البحث قدمنا القيم العددية للنتائج التي تم الحصول عليها و كذلك أشكال توضح العلاقة بين الحرارة الكلية للجسم و الزمن.