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SEMI-ANALYTICAL SPECTRAL SOLUTION FOR POISSON'S EQUATION USING DOUBLE ULTRASPHERICAL POLYNOMIALS

F.A.Hindi*

Abstract

We discuss the semi-analytical solution for Poisson's equation in two dimensions in a square, subject to the most general mixed boundary conditions, using double expansion in ultraspherical polynomials The extension of the method to the Heimholtz equation is also considered . Hence The differential equation with its boundary conditions is reduced to a system of linear equations This system was solved by using ECronecker matrix algebra . The method in its present form may be considered as a generalization of that of Doha[8]

1-Introduction

Historically, the name boundary value problem was attributed to only problems for which the partial differential equations are of are elliptic type. We extended the developments obtained to solve elliptic partial differential equations. On certain domains, such as rectangles, the so-called fast Poisson' S solvers tat are now widely available provide a rapid solutions of the standard five -points difference approximation to the partial differential equation (cf. Dorr[10] , Swarztrauber[18]
However the resolution of these methods is inherently limited by their algebraic convergence, i.e., provided that the solution of the continuous problem has continuous and bounded fourth order partial derivatives ,the maximum error of the discrete approximation with N+1 grid points in each direction decay as $\frac{1}{N^2}$

Because of this relatively slow rate of convergence the storage requirements can be quite severe and the computational time rather long for very accurate calculations A method for the solution of Poisson's equation in a rectangle based on the relation between the Fourier coefficients for the solution and these for the right-hand side is developed by Skollermo[17].The fast Fourier transform is used for the computation and its influence on the accuracy is studied. The method is shown to be second order accurate under certain general conditions on the smoothness of the solution. The accuracy is found to be limited by the lack of smoothness of the periodic extension of the inhomogeneous terms.

In recent years spectral methods have proven to be one way to manage the resolution problem, at least for smoothing the solution in simple geometries. Gottlieb and Orszag[13] solved Poisson's equation in a square ,by using spectral method. Haidvogel and Zang [15] used Chbyshev expansion technique to Poisson's equation on a square, with homogeneous Dirichlet boundary conditions. They have provided some comparisons between spectral and finite difference methods for solving Poisson's equation on a square. They presented some examples containing comer singularities indicating that although

*Girls college of education , Department of mathematics ,Jeddah, Saudi Arabia.

the spectral method had a finite rate of convergence, it had a lower absolute error than the finite difference methods. Horner[16] showed how a Chebyshev series in two variables can be used to solve elliptic partial differential equations of Laplace type with Dirichlet boundray conditions. A simple recurrence relation is obtained for the coefficients in the series solution. A double Chebyshev spectral method for the solution of Poisson's equation in a square subject to the most general mixed boundary conditions based on the relation between the Chebyshev coefficients for the solution and those for the right- hand side is developed by Doha[8].,

2. Some important results about functions of one and two variables in terms of ultraspherical polynomials.

Let the function f(x) be expressed as uniformly convergent series in the form

$$f(x)=\sum_{i=0}^{\infty} a_i C_i^{(\lambda)}(x) \qquad (1)$$

where a_i are constants. For the time being it is convenient to use the standardization $C_n^{(\lambda)}$ (1) =1, (n=0,1,...). This is not the usual standardization, but has the required properties that $C_n^{(0)}(x)$ is identical with the Chebyshev polynomial of the first kind $T_n(x)$. $C_n^{(\frac{1}{2})}(x)$ is the Legendre polynomial $P_n(x)$, and $C_n^{(1)}(x)$ is equal to $(1/(n+1))\, U_n(x)$, where $U_n(x)$ is the Chebyshev polynomial of the second kind. To find the finite sum consider the function

$$f(x)=\sum_{i=0}^{N} a_i C_i^{(\lambda)}(x) \qquad (2)$$

following Clenshaw[4] and by using the recurrence relation:

$$C_{n+1}^{(\lambda)}(x)+\alpha_n(x)C_n^{(\lambda)}(x)+\beta_n(x)C_{n-1}^{(\lambda)}(x)=0 \qquad (3)$$

where

$$\alpha_n(x)=-\frac{2(n+\lambda)}{(n+2\lambda)}.x \ , \ \beta_n(x)=\frac{n}{n+2\lambda} \qquad (4)$$

We construct the sequence $b_N, b_{N-1},\ldots,b_0$ such that $b_{N+1}=b_{N+2}=0$,

$$a_i=b_i+\alpha_i(x).b_{i+1}+\beta_{i+1}(x).b_{i+2} \quad , i=N,N-1,\ldots,0 \qquad (5)$$

Where $\alpha_i(x)$, $\beta_i(x)$ are as defined in (4).

After some manipulation, we can easily deduce that

$$f(x)=\begin{cases} b_0 & if \quad \lambda\neq 0 \\ \frac{1}{2}(b_0-b_2) & if \quad \lambda=0 \end{cases} \qquad (6)$$

Now, we can give analogous results for functions of two variables.
Let the function f (x ,y) be expressed as a uniformly convergent double series of ultraspherical polynomials in the form

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\cdot C_i^{(\lambda)}(x)C_j^{(\lambda)}(y) \qquad (7)$$

If $\lambda=0$ then equation (7) will be Chebyshev series expansion . Basu[2] refers to eq.(7) as a bivariate Chebyshev series expansion . Now if we write :

$$b_i = \sum_{j=0}^{m} a_{ij} C_j^{(\lambda)}(y) \tag{8}$$

$$c_j = \sum_{i=0}^{m} a_{ij} C_i^{(\lambda)}(x) \tag{9}$$

Then (7) can be splitted as

$$f(x,y) = \sum_{i=0}^{\infty} b_i \cdot C_i^{(\lambda)}(x) \tag{10}$$

$$f(x,y) = \sum_{j=0}^{\infty} c_j C_j^{(\lambda)}(y) \tag{11}$$

Thus, as in (10), let

$$S = S_{mn}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} C_i^{(\lambda)}(x) C_j^{(\lambda)}(y)$$

$$= \sum_{i=0}^{m} b_i C_i^{(\lambda)}(x)$$

where b_i as defined in (8). We construct a sequence of numbers

$$d_{i,n}, d_{i,n-1}, \ldots, d_{i,0}$$

such that

$$d_{i,n+2} = d_{i,n+1} = 0$$

$$a_{ij} = d_{ij} + \alpha_j(y) d_{i,j+1} + \beta_{j+1}(y) d_{i,j+2}, \quad j = n, n-1, \ldots, 0 \tag{12}$$

where $\alpha_j(y), \beta_j(y)$ are as defined in (4) . From the result of (6) , we deduce that

$$b_i = \begin{cases} d_{i,0} & \text{if} \quad \lambda \neq 0 \\ \frac{1}{2}(d_{i,0} - d_{i,2}) & \text{if} \quad \lambda = 0 \end{cases} \tag{13}$$

Now to find S we construct a new sequence of numbers $g_m, g_{m-1}, \ldots, g_0$ such that

$$g_{m+2} = g_{m+1} = 0 ,$$

$$b_i = g_i + \alpha_i(x) g_{i+1} + \beta_{i+1}(x) g_{i+2} , \tag{14}$$

$$i = m, m-1, \ldots, 0$$

where $\alpha_i(x)$,$\beta_i(x)$ are as previously defined in (4) . From (6) we have

$$S = S_{mn}(x,y) = \begin{cases} g_0 & \text{if} \quad \lambda \neq 0 \\ \frac{1}{2}(g_0 - g_2) & \text{if} \quad \lambda = 0 \end{cases} \tag{15}$$

Alternatively , from (11) , let

$$S = S_{mn}(x,y) = \sum_{j=0}^{n} c_j C_j^{(\lambda)}(y)$$

If we continue in the same manner we get the same result (eq.(15)) .

3. Discreption of the problem :

In this section we developed a method based on an expansion in ultraspherical polynomials for solving Poisson's equation in two space variables , namely

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y) \quad , \quad -1\le x,y\le 1 \tag{16}$$

subject to the most general inhomogeneous mixed boundary conditions

$$\left.\begin{aligned} \alpha_1 . u+\beta_1\frac{\partial u}{\partial x}&=\gamma_1(y) \quad , \quad x=-1\\ \alpha_2 . u+\beta_2\frac{\partial u}{\partial x}&=\gamma_2(y) \quad , \quad x=1\end{aligned}\right] \quad , \quad -1\le y\le 1 \tag{17}$$

$$\left.\begin{aligned} \alpha_3 . u+\beta_3\frac{\partial u}{\partial y}&=\gamma_3(x) \quad , \quad y=-1\\ \alpha_4 . u+\beta_4\frac{\partial u}{\partial y}&=\gamma_4(x) \quad , \quad y=1\end{aligned}\right] \quad , \quad -1\le x\le 1 \tag{18}$$

Assuming that the solution to the above problem (16) can be expressed in a uniformly convergent double ultraspherical series expansion

$$U(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}C_m^{(\lambda)}(x)C_n^{(\lambda)}(y) \tag{19}$$

Throughout this work we assumed that there is no discontinuity between the boundary conditions at the four corners of the domain of solution . We also assumed that f (x ,y) has known ultraspherical series expansion

$$f(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f_{mn}C_m^{(\lambda)}(x)C_n^{(\lambda)}(y) \tag{20}$$

which is uniformly convergent in $-1\le x,y\le 1$. Then it follows that the solution of (16) has a double series expansion of the form (19) and the solution is free of discontinuities . Now let us assume the following expansions

$$\frac{\partial^2 u}{\partial x^2}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}^{(2,0)}C_m^{(\lambda)}(x)C_n^{(\lambda)}(y) \tag{21}$$

$$\frac{\partial^2 u}{\partial y^2}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}^{(0,2)}C_m^{(\lambda)}(x)C_n^{(\lambda)}(y) \tag{22}$$

where $a_{mn}^{(p,q)}$ denote the ultraspherical coefficients of $\frac{\partial^{p+q}u}{\partial x^p\partial y^q}$ .Also, assuming that

$$\gamma_i(y)=\sum_{n=0}^{\infty}\gamma_n^{(i)}C_n^{(\lambda)}(y) \quad , \quad i=1,2 \tag{23}$$

$$\gamma_i(x)=\sum_{m=0}^{\infty}\gamma_m^{(i)}C_{mn}^{(\lambda)}(x) \quad , \quad i=3,4 \tag{24}$$

3.1 Derivation of the method of solution :

A two -dimensional ultraspherical expansion produced by the tensor product choice

$$\phi_{mn}(x,y)=C_m^{(\lambda)}(x)C_n^{(\lambda)}(y)\begin{cases}m=0,1,2,\dots,M\\ n=0,1,2,\dots,N\end{cases} \tag{25}$$

where $C_m^{(\lambda)}(x)$ and $C_n^{(\lambda)}(y)$ are the ultraspherical polynomials of degrees m and n in x and y respectively . The truncated ultraspherical expansion is

$$U(x,y) \approx u(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn} C_m^{(\lambda)}(x) C_n^{(\lambda)}(y) \tag{26}$$

Note that the trial functions do not satisfy the boundary conditions individually . Thus, it is necessary to have weighted residual conditions for both the partial differential equation (16) and the boundary conditions (17) and (18) . Two sets of test functions are used . For the partial differential equation , they are

$$\psi_{mn}(x,y) = Q_m(x) Q_n(y) \begin{cases} m = 0,1,2,\ldots,M-2 \\ n = 0,1,2,\ldots,N-2 \end{cases} \tag{27}$$

where

$$Q_m(x) = \frac{C_m^{(\lambda)}(x)}{h_m} \quad , \quad Q_n(y) = \frac{C_n^{(\lambda)}(y)}{h_n} \quad ,$$
$$h_m = \frac{(m+\lambda).\Gamma(m+2\lambda)}{\pi .2^{1-2\lambda} \quad m!}\left[\frac{\Gamma(\lambda)}{\Gamma(2\lambda)}\right]^2 \tag{28}$$

For the boundary conditions , they are

$$\eta_n^{(i)}(y) = Q_n(y) \quad , \quad i = 1,2 \ ; \ n = 0,1,\ldots,N \tag{29}$$

$$\eta_m^{(i)}(x) = Q_m(x) \quad , \quad i = 3,4 \ ; \ m = 0,1,\ldots,M \tag{30}$$

The method of weighted residual conditions are

$$\int_{-1}^{1} dy \int_{-1}^{1} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f(x,y)\psi_{mn}(x,y)\right)dx = 0 \begin{bmatrix} m = 0,1,2,\ldots,M-2 \\ n = 0,1,2,\ldots,N-2 \end{bmatrix} \tag{31}$$

and

$$\left.\begin{aligned} &\int_{-1}^{1}(\alpha_1 u + \beta_1 \frac{\partial u}{\partial x} - \gamma_1(y))\eta_n^{(1)}(y).dy = 0 \\ &\int_{-1}^{1}(\alpha_2 u + \beta_2 \frac{\partial u}{\partial x} - \gamma_2(y))\eta_n^{(2)}(y).dy = 0 \end{aligned}\right], \ n = 0,1,\ldots,N \tag{32}$$

$$\left.\begin{aligned} &\int_{-1}^{1}(\alpha_3 u + \beta_3 \frac{\partial u}{\partial y} - \gamma_3(x))\eta_m^{(3)}(x).dx = 0 \\ &\int_{-1}^{1}(\alpha_4 u + \beta_4 \frac{\partial u}{\partial y} - \gamma_4(x))\eta_m^{(4)}(x).dx = 0 \end{aligned}\right], \ m = 0,1,\ldots,M \tag{33}$$

Four of the conditions in (32) and (33) are linearly dependent upon the others ; in effect the boundary conditions at each of the four corner points have been applied twice . The above integrals (31) ,(32) and (33) may be performed analytically .The result is

$$a_{mn}^{(2,0)} + a_{mn}^{(0,2)} = f_{mn} \begin{cases} m = 0,1,2,\ldots,M-2 \\ n = 0,1,2,\ldots,N-2 \end{cases} \tag{34}$$

$$\left.\begin{aligned} &\sum_{m=0}^{M}(-1)^m\left[\alpha_1 - \frac{m(m+2\lambda)}{1+2\lambda}\beta_1\right] a_{mn} = \gamma_n^{(1)} \\ &\sum_{m=0}^{M}\left[\alpha_2 + \frac{m(m+2\lambda)}{1+2\lambda}\beta_2\right] a_{mn} = \gamma_n^{(2)} \end{aligned}\right\} n = 0,1,\ldots,N \tag{35}$$

$$\left.\begin{array}{l}\sum_{n=0}^{N}(-1)^n\left[\alpha_3-\frac{n(n+2\lambda)}{1+2\lambda}\beta_3\right]a_{mn}=\gamma_m^{(3)}\\ \sum_{n=0}^{N}\left[\alpha_4+\frac{n(n+2\lambda)}{1+2\lambda}\beta_{42}\right]a_{mn}=\gamma_m^{(4)}\end{array}\right\}m=0,1,\dots,M \tag{36}$$

where

$$f_{mn}=\int_{-1}^{1}(1-y^2)^{\lambda-\frac{1}{2}}dy.\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}f(x,y).\psi_{mn}(x,y).dx \tag{37}$$

$$a_{mn}^{(2,0)}=\int_{-1}^{1}(1-y^2)^{\lambda-\frac{1}{2}}dy.\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}.\frac{\partial^2 u}{\partial x^2}.\psi_{mn}(x,y).dx \tag{38}$$

$$a_{mn}^{(0,2)}=\int_{-1}^{1}(1-y^2)^{\lambda-\frac{1}{2}}dy.\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}.\frac{\partial^2 u}{\partial y^2}.\psi_{mn}(x,y).dx \tag{39}$$

$$\gamma_n^{(i)}=\int_{-1}^{1}(1-y^2)^{\lambda-\frac{1}{2}}\gamma_i(y)\eta_n^{(i)}(y).dy \qquad i=1,2 \tag{40}$$

$$\gamma_m^{(i)}=\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\gamma_i(x)\eta_m^{(i)}(x).dx \qquad i=3,4 \tag{41}$$

Now, the following two expressions due to Doha[9] are of fundamental importance in the derivation of the method of this article .These two expressions are :

$$a_{mn}^{(p-1,q)}=\frac{m+2\lambda-1}{2m(m+\lambda-1)}a_{m-1,n}^{(p,q)}-\frac{m+1}{2(m+\lambda+1).(m+2\lambda)}a_{m+1,n}^{(p,q)} \qquad m,p\geq 1 \tag{42}$$

$$a_{mn}^{(p,q-1)}=\frac{n+2\lambda-1}{2n(n+\lambda-1)}a_{m,n-1}^{(p,q)}-\frac{n+1}{2(n+\lambda+1).(n+2\lambda)}a_{m,n+1}^{(p,q)} \qquad n,q\geq 1 \tag{43}$$

If we replace m by (m-1) and (m+1) resp. in (34) ,we get

$$a_{m-1,n}^{(2,0)}+a_{m-1,n}^{(0,2)}=f_{m-1,n} \tag{44}$$

$$a_{m+1,n}^{(2,0)}+a_{m+1,n}^{(0,2)}=f_{m+1,n} \tag{45}$$

multiplying both sides of (44) and (45) by $\frac{m+1}{2(m+\lambda+1).(m+2\lambda)}$, $\frac{m+2\lambda-1}{2m(m+\lambda-1)}$ respectively,we obtain

$$\frac{m+2\lambda-1}{2m(m+\lambda-1)}\left[a_{m-1,n}^{(2,0)}+a_{m-1,n}^{(0,2)}\right]=\frac{m+2\lambda-1}{2m(m+\lambda-1)}f_{m-1,n} \tag{46}$$

$$\frac{m+1}{2(m+\lambda+1).(m+2\lambda)}\left[a_{m+1,n}^{(2,0)}+a_{m+1,n}^{(0,2)}\right]=\frac{m+1}{2(m+\lambda+1).(m+2\lambda)}f_{m+1,n} \tag{47}$$

subtracting (47) from (46) and making use of (42), gives immediately

$$a_{mn}^{(1,0)}+\left(\frac{m+2\lambda-1}{2m(m+\lambda-1)}a_{m-1,n}^{(0,2)}-\frac{m+1}{2(m+\lambda+1).(m+2\lambda)}a_{m+1,n}^{(0,2)}\right)=\frac{m+2\lambda-1}{2m(m+\lambda-1)}f_{m-1,n}-\frac{m+1}{2(m+\lambda+1).(m+2\lambda)}f_{m+1,n} \tag{48}$$

also if we replace m by (m-1) and (m+1) resp. in (48) ,we obtain

$$a_{m-1,n}^{(1,0)}+\left(\frac{m+2\lambda-2}{2(m-1)(m+\lambda-2)}a_{m-2,n}^{(0,2)}-\frac{m}{2(m+\lambda).(m+2\lambda-1)}a_{mn}^{(0,2)}\right)$$
$$=\frac{m+2\lambda-2}{2(m-1)(m+\lambda-2)}f_{m-2,n}-\frac{m}{2(m+\lambda).(m+2\lambda-1)}f_{mn} \qquad (49)$$

$$a_{m+1,n}^{(1,0)}+\left(\frac{m+2\lambda}{2(m+1)(m+\lambda)}a_{mn}^{(0,2)}-\frac{m+2}{2(m+\lambda+2).(m+2\lambda+1)}a_{m+2,n}^{(0,2)}\right)$$
$$=\frac{m+2\lambda}{2(m+1)(m+\lambda)}f_{mn}-\frac{m+2}{2(m+\lambda+2).(m+2\lambda+1)}f_{m+2,n} \qquad (50)$$

Multiplying both sides of (49) and (50) by $\frac{m+2\lambda-1}{2m(m+\lambda-1)}$, $\frac{m+1}{2(m+\lambda+1).(m+2\lambda)}$ respectively ; then subtracting the new obtained equations and making use of (42) and often a lengthy manipulation , we get

$$a_{mn}+\frac{(m+2\lambda-1)(m+2\lambda-2)}{4m(m-1)(m+\lambda-1)(m+\lambda-2)}a_{m-2,n}^{(0,2)}-\frac{1}{2(m+\lambda-1)(m+\lambda+1)}a_{mn}^{(0,2)}$$
$$+\frac{(m+1)(m+2)}{4(m+2\lambda)(m+2\lambda+1)(m+\lambda+1)(m+\lambda+2)}a_{m+2,n}^{(0,2)}=$$
$$\frac{(m+2\lambda-1)(m+2\lambda-2)}{4m(m-1)(m+\lambda-1)(m+\lambda-2)}f_{m-2,n}-\frac{1}{2(m+\lambda-1)(m+\lambda+1)}f_{mn} \qquad (51)$$
$$+\frac{(m+1)(m+2)}{4(m+2\lambda)(m+2\lambda+1)(m+\lambda+1)(m+\lambda+2)}f_{m+2,n}$$

Equation (51) may be rewritten in the matrix form as:

$$a_{mn}+\sum_{i=0}^{M-2}A_{mi}a_{in}^{(0,2)}=\sum_{i=0}^{M-2}A_{mi}f_{in} \qquad n=0,1,2,\ldots,N-2 \qquad (52)$$

where

$$A_{mi}=\begin{cases}\frac{(m+2\lambda-1)(m+2\lambda-2)}{4m(m-1)(m+\lambda-1)(m+\lambda-2)} & i=m-2 \ (\textit{to be doubled if } \lambda=0\ ,\ m=2)\\ \frac{-1}{2(m+\lambda-1)(m+\lambda+1)} & i=m\\ \frac{(m+1)(m+2)}{4(m+2\lambda)(m+2\lambda+1)(m+\lambda+1)(m+\lambda+2)} & i=m+2\\ 0 & \textit{unless } i=m\ ,m\pm 2\end{cases} \qquad (53)$$

Now if we use the second relation (43) instead of (42) , with p=0 ,q=2, then after a rather tedious lengthy manipulation , equation (52) may be written in the compact matrix form

$$\sum_{i=0}^{M}A_{mi}a_{in}+\sum_{j=0}^{N}a_{mj}B_{jn}=\sum_{i=0}^{M}\sum_{j=0}^{N}A_{mi}f_{ij}B_{jn} \qquad m=2,3,\ldots,M \quad , \quad n=2,3,\ldots,N \quad (54)$$

where

$B_{jn}=A_{nj}$

It is worthy to mention here that Doha [9] defined a related set of coefficients

$b_{mn}^{(p,q)}$ by writing

$$a_{mn}^{(p,q)} = \frac{(m+\lambda)(n+\lambda)\Gamma(m+2\lambda)\Gamma(n+2\lambda)}{m!\quad n!} b_{mn}^{(p,q)} \quad m,n \geq 0 \quad , \quad p,q = 0,1,... \tag{55}$$

Equations (42) and (43) take the simpler forms

$$b_{m-1,n}^{(p,q)} - b_{m+1,n}^{(p,q)} = 2(m+\lambda) b_{mn}^{(p-1,q)} \qquad m,p \geq 1 \tag{56}$$

$$b_{m,n-1}^{(p,q)} - b_{m,n+1}^{(p,q)} = 2(n+\lambda) b_{mn}^{(p,q-1)} \qquad n,q \geq 1 \tag{57}$$

Equation (34) also becomes

$$b_{mn}^{(2,0)} + b_{m,n}^{(0,2)} = \frac{m!\quad n!}{(m+\lambda)(n+\lambda)\Gamma(m+2\lambda)\Gamma(n+2\lambda)} f_{mn} \tag{58}$$

If we begin with (58) using the same method but applied to (56) and (57), we obtain equation (54), but with little effort.

Now, by letting m and n range from 2 to M and 2 to N respectively, we obtain a total of (M-1) (N-1) equations from the equation (54). The remaining (M +1) (N +1) - (M -1) (N -1) = 2M + 2N equations required for finding the coefficients $\{a_{mn}\}$, m= 0 ,1, 2, ...,M, n= 0,1, 2, ..., N are obtained from the boundary conditions .

3.2 Utilizing the boundary conditions :

Rewrite the equations (35) ,(36) after some manipulation , in the form

$$\left.\begin{array}{l} a_{0n} + \sum_{m=2}^{M} \mu_m a_{mn} = g_n \\ a_{1n} + \sum_{m=2}^{M} v_m a_{mn} = h_n \end{array}\right] \quad n = 0,1,...,N \tag{59}$$

$$\left.\begin{array}{l} a_{m0} + \sum_{n=2}^{N} u_n a_{mn} = r_m \\ a_{m1} + \sum_{n=2}^{N} v_n a_{mn} = s_m \end{array}\right] \quad m = 0,1,...,M \tag{60}$$

where

$$\delta_1 = 2\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$$

$$\mu_m = \frac{(\alpha_1 - \beta_1)\left[\alpha_2 + \frac{m(m+2\lambda)}{1+2\lambda}\beta_2\right] + (-1)^m(\alpha_2 + \beta_2)\left[\alpha_1 - \frac{m(m+2\lambda)}{1+2\lambda}\beta_1\right]}{\delta_1}$$

$$g_n = \left[(\alpha_2 + \beta_2)\gamma_n^{(1)} + (\alpha_1 - \beta_1)\gamma_n^{(2)}\right] / \delta_1$$

$$v_m = \frac{\alpha_1\left[\alpha_2 + \frac{m(m+2\lambda)}{1+2\lambda}\beta_2\right] - (-1)^m \alpha_2\left[\alpha_1 - \frac{m(m+2\lambda)}{1+2\lambda}\beta_1\right]}{\delta_1}$$

$$h_n = \left[\alpha_1 \cdot \gamma_n^{(2)} - \alpha_2 \cdot \gamma_n^{(1)}\right] / \delta_1$$

$$\delta_2 = 2\alpha_3\alpha_4 + \alpha_3\beta_4 - \alpha_4\beta_3 \neq 0$$

$$u_n = \frac{(\alpha_3 - \beta_3)\left[\alpha_4 + \frac{n(n+2\lambda)}{1+2\lambda}\beta_4\right] + (-1)^n(\alpha_4 + \beta_4)\left[\alpha_3 - \frac{n(n+2\lambda)}{1+2\lambda}\beta_3\right]}{\delta_2}$$

$$r_m = \left[(\alpha_4 + \beta_4)\gamma_m^{(3)} + (\alpha_3 - \beta_3)\gamma_m^{(4)}\right] / \delta_2$$

$$v_n = \frac{\alpha_3\left[\alpha_4 + \frac{n(n+2\lambda)}{1+2\lambda}\beta_4\right] - (-1)^n \alpha_4\left[\alpha_3 - \frac{n(n+2\lambda)}{1+2\lambda}\beta_3\right]}{\delta_2}$$

$$s_m = \left[\alpha_3 \cdot \gamma_m^{(4)} + \alpha_4 \cdot \gamma_m^{(3)}\right] / \delta_2$$

Note that the boundary conditions (59) and (60) are not all linearly independent ,there exist four linear relation among them . Equations (59) and (60) may be used to eliminate $a_{0n}, a_{1n}, a_{m0},$ *and* a_{m1} from the left hand side of equation (54) to give

$$A_{m0}g_n + A_{m1}h_n + B_{0n}r_m + B_{1n}s_m + \sum_{i=2}^{M}\left[A_{mi} - A_{m0}\mu_i - A_{m1}v_i\right]a_{in}$$
$$+\sum_{j=2}^{N}a_{mj}\left[B_{jn} - B_{0n}u_j - B_{1n}v_j\right] = \sum_{i=0}^{M}\sum_{j=0}^{N}A_{mi}f_{ij}B_{jn} \quad ; \; 2 \le m \le M \;,\; 2 \le n \le N$$

which in turn may be written as :

$$\sum_{i=2}^{M}C_{mi}a_{in} + \sum_{j=2}^{N}a_{mj}D_{jn} = b_{mn} \quad ; 2 \le m \le M \;,\; 2 \le n \le N \tag{61}$$

where

$$\left.\begin{array}{ll} C_{2i} = A_{2i} - \mu_i A_{20} & D_{j2} = B_{j2} - B_{02}u_j \\ C_{3i} = A_{3i} - v_i A_{31} & ; D_{j3} = B_{j3} - B_{13}v_j \\ C_{mi} = A_{mi} \quad m \ge 4 & D_{jn} = B_{jn} \quad n \ge 4 \end{array}\right\} \tag{62}$$

and

$$b_{mn} = \sum_{i=0}^{M}\sum_{j=0}^{N}A_{mi}f_{ij}B_{jn} - (A_{m0}g_n + A_{m1}h_n + B_{0n}r_m + B_{1n}s_m) \tag{63}$$

let A be the (M-1) (N -1) matrix with (m,n)th element $a_{m+1,n+1}$,$1 \le m \le M-1$,$1 \le n \le N-1$

Then equation (61) may be written in the matrix form

$$C\,A + A\,D = B \tag{64}$$

where

C is an (M-1) × (M-1) matrix with (m,n)th element $c_{m+1,n+1}$; D is an (N -1) × (N -1) matrix with (m,n)th element $d_{m+1,n+1}$; and B is an (M -1) × (N -1) matrix with (m,n)th element $b_{m+1,n+1}$. It is worthy to note that the matrix C represents the ultraspherical approximation to $\frac{\partial^2}{\partial x^2}$ with the nonhomogeneous mixed boundray conditions ,i.e., C A represents the first term in equation (54) with the boundary conditions of equation (59) used to eliminate a_{0n} and a_{1n} for $0 \le n \le N$. The y-derivative in equation (54) appears above in the form A D.

The matrix equation (64) represents a system of linear algebraic equations in the rectangular matrix A of the type (M-1)×(N-1) consisting of the element a_{ij} $(2 \le i \ge M, 2 \le j \le N)$. A method using the kronecker matrix algebra for solving such system is introduced in the next article.

3.3 Solving the system of equation (64)

Let the kronecker product of the two matrices C and D be defined by :

$$C \otimes D = [c_{ij}.D] \qquad (i,j = 2,\ldots,M)$$

and their kronecker sum as

$$C \oplus D = C \otimes I_{N-1} + I_{M-1} \otimes D$$

where I_{M-1} and I_{N-1} are the identity matrices of order (M -1) and (N -1) respectivèly .

Now, introducing the so called block vectors $\underline{a} = [a_{-2}, a_{-3}, \ldots, a_{-N}]^T, \underline{b} = [b_{-2}, b_{-3}, \ldots, b_{-N}]^T$ consisting of the columns of the matrices A and B respectively,where:

$A = [a_{-2} a_{-3} \ldots a_{-N}], B = [b_{-2} b_{-3} \ldots b_{-N}]$ then it is easy to prove that the system (64) is equivalent to the following system

$$G\ \underline{a} = \underline{b} \tag{65}$$

where the coefficient matrix G is equal to the kronecker sum $D^T \oplus C$,where D^T denotes the transpose of D. More detailed description of this algebra can be found in Graham [14] .

4. An alternative contributed method of solution

If we assume that U(x ,y) and its partial derivatives $U^{(p,q)}(x,y) = \frac{\partial^{p+q} U(x,y)}{\partial x^p \partial y^q}$ have the double ultraspherical series expansions given by (19) ,(21) and(22). Then Doha [9] proved that

$$a_{mn}^{(p,q)} = \frac{2^p \Gamma(m+\lambda)\Gamma(m+2\lambda)}{(p-1)!\ m!} \sum_{i=1}^{\infty} \frac{(i+p-2)!\,\Gamma(m+i+p+\lambda-1)}{(i-1)!\,\Gamma(m+i+\lambda)} * \frac{(m+2i+p-2)!}{\Gamma(m+2i+p+2\lambda-2)} a_{m+2i+p-2,n}^{(0,q)} \quad , \quad p \ge 1 \tag{66}$$

and

$$a_{mn}^{(p,q)} = \frac{2^q (n+\lambda)\Gamma(n+2\lambda)}{(q-1)!\ n!} \sum_{j=1}^{\infty} \frac{(j+q-2)!\Gamma(n+j+q+\lambda-1)}{(j-1)!\Gamma(n+j+\lambda)} * \frac{(n+2j+q-2)!}{\Gamma(n+2j+q+2\lambda-2)} a_{m,n+2j+q-2}^{(p,0)} \quad , \quad q \geq 1 \tag{67}$$

Now , it can be easily shown from (66) and (67) that

$$a_{mn}^{(2,0)} = \frac{(m+\lambda)\Gamma(m+2\lambda)}{m!} \sum_{\substack{i=m+2 \\ (i-m)even}}^{\infty} \frac{i!\left[(i+\lambda)^2-(m+\lambda)^2\right]}{\Gamma(i+2\lambda)} a_{in} \tag{68}$$

$$a_{mn}^{(0,2)} = \frac{(n+\lambda)\Gamma(n+2\lambda)}{n!} \sum_{\substack{j=n+2 \\ (j-n)even}}^{\infty} \frac{j!\left[(j+\lambda)^2-(n+\lambda)^2\right]}{\Gamma(j+2\lambda)} a_{mj} \tag{69}$$

It follows from (67) and (68) that the partial differential equation (16)- (utilizing equation (34))- is equivalent to

$$\sum_{i=0}^{M} K_{mi}\, a_{in} + \sum_{j=0}^{N} a_{mj}\, L_{jn} = f_{mn} \qquad ,0 \leq n \leq N \quad , \quad 0 \leq m \leq M \tag{70}$$

where

$$K_{mi} = \frac{i!\ (m+\lambda)\Gamma(m+2\lambda)\left[(i+\lambda)^2-(m+\lambda)^2\right]}{m!\ \Gamma(i+2\lambda)} \qquad ,i \geq m+2,(i-m)even \tag{71}$$

$$L_{jn} = \frac{j!\ (n+\lambda)\Gamma(n+2\lambda)\left[(j+\lambda)^2-(n+\lambda)^2\right]}{n!\ \Gamma(j+2\lambda)} \qquad ,j \geq n+2,(j-n)even \tag{72}$$

and zero otherwise.

Now , we propose that a_{in} and a_{mj} can be neglected for $m \geq M+1$, $n \geq N+1$ and to eliminating $a_{i,N-1}, a_{i,N}, a_{M-1,j}$ *and* $a_{M,j}$ by making use of the boundary conditions(35)and (36) .The Conditions (35) and (36) after some lengthy manipulation ,may be put in the form

$$\left.\begin{aligned} a_{M-1,n} + \sum_{m=0}^{M-2} \mu'_m a_{mn} &= g'_n \\ a_{M,n} + \sum_{m=0}^{M-2} v'_m a_{mn} &= h'_n \end{aligned}\right\} \quad n = 0,1,\ldots,N \tag{73}$$

$$\left.\begin{aligned} a_{m,N-1} + \sum_{n=0}^{N-2} u'_n a_{mn} &= r'_m \\ a_{m,N} + \sum_{n=0}^{N-2} v'_n a_{mn} &= s'_n \end{aligned}\right\} \quad m = 0,1,\ldots,M \tag{74}$$

where

$$\delta'_1 = (-1)^M \Bigg\{ \left[\alpha_1 - \frac{M(M+2\lambda)}{1+2\lambda}\beta_1\right]\left[\alpha_2 + \frac{(M-1)(M+2\lambda-1)}{1+2\lambda}\beta_2\right] + \left[\alpha_2 + \frac{M(M+2\lambda)}{1+2\lambda}\beta_2\right]\left[\alpha_1 - \frac{(M-1)(M+2\lambda-1)}{1+2\lambda}\beta_1\right] \Bigg\} \neq 0$$

$$\mu'_m = \Bigg\{ (-1)^M \left[c_1 - \frac{M(M+2\lambda)}{1+2\lambda}\beta_1\right]\left[\alpha_2 + \frac{m(m+2\lambda)}{1+2\lambda}\beta_2\right] - (-1)^m \left[\alpha_1 - \frac{m(m+2\lambda)}{1+2\lambda}\beta_1\right]\left[\alpha_2 + \frac{M(M+2\lambda)}{1+2\lambda}\beta_2\right] \Bigg\} \Big/ \delta'_1$$

$$g'_n = \frac{(-1)^M \left[\alpha_1 - \frac{M(M+2\lambda)}{1+2\lambda}\beta_1\right]\gamma_n^{(2)} - \left[\alpha_2 + \frac{M(M+2\lambda)}{1+2\lambda}\beta_2\right]\gamma_n^{(1)}}{\delta'_1}$$

$$\nu'_m = \Bigg\{ (-1)^m \left[\alpha_1 - \frac{m(m+2\lambda)}{1+2\lambda}\beta_1\right]\left[\alpha_2 + \frac{(M-1)(M+2\lambda-1)}{1+2\lambda}\beta_2\right] + (-1)^M \left[\alpha_1 - \frac{(M-1)(M+2\lambda-1)}{1+2\lambda}\beta_1\right]\left[\alpha_2 + \frac{m(m+2\lambda)}{1+2\lambda}\beta_2\right] \Bigg\} \Big/ \delta'_1$$

$$h'_n = \frac{\left[\alpha_2 + \frac{(M-1)(M+2\lambda-1)}{1+2\lambda}\beta_2\right]\gamma_n^{(1)} + (-1)^M \left[\alpha_1 - \frac{(M-1)(M+2\lambda-1)}{1+2\lambda}\beta_1\right]\gamma_n^{(2)}}{\delta'_1}$$

$$\delta'_2 = (-1)^N \Bigg\{ \left[\alpha_3 - \frac{N(N+2\lambda)}{1+2\lambda}\beta_3\right]\left[\alpha_4 + \frac{(N-1)(N+2\lambda-1)}{1+2\lambda}\beta_4\right] + \left[\alpha_4 + \frac{N(N+2\lambda)}{1+2\lambda}\beta_4\right]\left[\alpha_3 - \frac{(N-1)(N+2\lambda-1)}{1+2\lambda}\beta_3\right] \Bigg\} \neq 0$$

$$u'_n = \Bigg\{ (-1)^N \left[\alpha_3 - \frac{N(N+2\lambda)}{1+2\lambda}\beta_3\right]\left[\alpha_4 + \frac{n(n+2\lambda)}{1+2\lambda}\beta_4\right] - (-1)^n \left[\alpha_3 - \frac{n(n+2\lambda)}{1+2\lambda}\beta_3\right]\left[\alpha_4 + \frac{N(N+2\lambda)}{1+2\lambda}\beta_4\right] \Bigg\} \Big/ \delta'_2$$

$$r'_m = \frac{(-1)^N\left[\alpha_3 - \frac{N(N+2\lambda)}{1+2\lambda}\beta_3\right]\gamma_m^{(4)} - \left[\alpha_4 + \frac{N(N+2\lambda)}{1+2\lambda}\beta_4\right]\gamma_m^{(3)}}{\delta'_2}$$

$$v'_2 = \left\{(-1)^n\left[\alpha_3 - \frac{n(n+2\lambda)}{1+2\lambda}\beta_3\right]\left[\alpha_4 + \frac{(N-1)(N+2\lambda-1)}{1+2\lambda}\beta_4\right] + (-1)^N\left[\alpha_3 - \frac{(N-1)(N+2\lambda-1)}{1+2\lambda}\beta_3\right]\left[\alpha_4 + \frac{n(n+2\lambda)}{1+2\lambda}\beta_4\right]\right\} / \delta'_2$$

$$s'_m = \frac{\left[\alpha_4 + \frac{(N-1)(N+2\lambda-1)}{1+2\lambda}\beta_4\right]\gamma_m^{(3)} + (-1)^N\left[\alpha_3 - \frac{(N-1)(N+2\lambda-1)}{1+2\lambda}\beta_3\right]\gamma_m^{(4)}}{\delta'_2}$$

using (71) and (72) to eliminate $a_{M-1,n}, a_{M,n}, a_{m,N-1}, a_{m,N}$ from the finite form (70) leads to

$$\sum_{i=0}^{M-2} H_{mi} a_{in} + \sum_{j=0}^{N-2} a_{mj} T_{jn} = b'_{mn} \qquad (0 \le m \le M-2 \ , \ 0 \le n \le N-2) \qquad (73)$$

where

$$H_{mi} = K_{mi} - K_{m,M-1}\mu'_i - K_{m,M}v'_i \qquad (74)$$

$$T_{jn} = L_{jn} - L_{N-1,n}u'_j - L_{N,n}v'_j \qquad (75)$$

$$b'_{mn} = f_{mn} - (K_{m,M-1}g'_n + K_{m,M}h'_n + L_{N-1,n}r'_m + L_{N,n}s'_m \qquad (76)$$

Equation (73) may be written in the matrix form

$$H\,A + A\,T = B' \qquad (77)$$

where A is (M-1)$\times$(N -1) matrix with (m,n)th element a_{mn} $,0 \le m \le M-2 \ ,0 \le n \le N-2$.

This equation has the same method of solution as the matrix equation (64).

This alternative approach leads to an equation which is similar to equation(64). There is some computational advantages in this approach , equation (77) throws some light on the structure of the matrices H and T which were not previously apparent .Note that many of the elements of H and the transpose of T ,including all those on and below the main diagonals are zero. Note also that although the method of this article is computationally simpler than that of article 3 ,it is mathematically equivalent and will produce identical results , which is a big advantage to our proposal .

5. Extension of the methods to solve Helmholtz equation

To extend the ultraspherical expansion methods of articles 3 and 4 to handle Helmholtz equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\alpha\, U = f(x,y) \tag{78}$$

For constant α subject to the more general inhomogeneous mixed boundary conditions (17) and (18) . This needs a complicated modifications of (61) and a simple one for (73), by adding to both sides, terms which reflect the steps of their derivations applied to the coefficients of U(x, y).

Now ,by applying the procedure of article 3 to the equation (78) we get

$$\sum_{i=2}^{M} C_{mi}a_{in}+\sum_{j=2}^{N} a_{mj}D_{jn}+\alpha\sum_{i=0}^{M}\sum_{j=0}^{N} A_{mi}a_{ij}B_{jn}=b_{mn} \tag{79}$$

where C_{mi}, D_{jn} and b_{mn} are previously given as in (62) and (63) respectively.The difference between (79) and (61)apart from the parameter α is the additional term , $\sum_{i=0}^{M}\sum_{j=0}^{N} A_{mi}a_{ij}B_{jn}$,which may be written in the form

$$\begin{aligned}\sum_{i=0}^{M}\sum_{j=0}^{N} A_{mi}a_{ij}B_{jn} &= A_{m0}B_{0n}a_{00}+A_{m0}B_{1n}a_{01}+A_{m1}B_{0n}a_{10}+A_{m1}B_{1n}a_{11}+A_{m0}\sum_{j=2}^{N}a_{0j}B_{jn}\\ &+A_{m1}\sum_{j=2}^{N}a_{1j}B_{jn}+B_{0n}\sum_{i=2}^{M}A_{mi}a_{i0}+B_{1n}\sum_{i=1}^{M}A_{mi}a_{i1}+\sum_{i=2}^{M}\sum_{j=2}^{N}A_{mi}a_{ij}B_{jn}\end{aligned} \tag{80}$$

Equations (59) and (60) again may be used to eliminate the coefficients $a_{00}, a_{01}, a_{10}, a_{11}, a_{0j}, a_{1j}, a_{i0}$ *and* a_{i1} from the right-hand side of (80) .This process involves very tedious and lengthy manipulation. Performing this elimination left us with

$$\begin{aligned}\sum_{i=0}^{M}\sum_{j=0}^{N} A_{mi}a_{ij}B_{jn} &=(A_{m0}g_0+A_{m1}h_0)B_{0n}+(A_{m0}g_1+A_{m1}h_1)B_{1n}+\sum_{j=2}^{N}(A_{m0}g_j+A_{m1}h_j)B_{jn}\\ &+\sum_{i=2}^{M}A_{mi}(r_iB_{0n}+s_iB_{1n})-\sum_{i=2}^{M}(A_{m0}\mu_i+A_{m1}v_i)(B_{0n}r_i+B_{1n}s_i)\\ &+\sum_{i=2}^{M}\sum_{j=2}^{N}C_{mi}a_{ij}D_{jn} \quad ; \quad 2\le m\le M \quad , \quad 2\le n\le N\end{aligned} \tag{81}$$

substitution from (81) into (79) gives

$$\sum_{i=2}^{M} C_{mi}a_{in}+\sum_{j=2}^{N} a_{mj}D_{jn}+\alpha\sum_{i=2}^{M}\sum_{j=2}^{N} C_{mi}a_{ij}D_{jn}=e_{mn} \quad , 2\le m\le M\ , 2\le n\le N \tag{82}$$

where

$$e_{mn} = b_{mn} - \alpha\Big[\big(A_{m0}g_0 + A_{m1}h_0\big)B_{0n} + \big(A_{m0}g_1 + A_{m1}h_1\big)B_{1n} + \sum_{j=2}^{N}(A_{m0}g_j + A_{m1}h_j)$$
$$+ \sum_{i=2}^{M} A_{mi}(B_{0n}r_i + B_{1n}s_i) - \sum_{i=2}^{M}(A_{m0}\mu_i + A_{m1}v_i)(B_{0n}r_i + B_{1n}s_i)\Big];\ 2 \le m \le M, 2 \le n \le N \tag{83}$$

Equation (82)may be written in the matrix form

$$C\,A + A\,D + \alpha\,C\,A\,D = E \qquad ;2 \le m \le M,\ 2 \le n \le N \tag{84}$$

where E is an $(M-1)\times(N-1)$ matrix with (m,n)th element e_{mn} . The matrix equation (84) represents a system of linear equations in the rectangular matrix A and has the same method of solution like that of equation (64). If we apply the alternative method of article 4 to equation (78) , we get the following modified form of (73) as

$$\sum_{i=0}^{M-2} H_{mi}a_{in} + \sum_{j=0}^{N-2} a_{mj}T_{jn} + \alpha . a_{mn} = b'_{mn};\ 2 \le m \le M-2\ ,\ 2 \le n \le N-2 \tag{85}$$

where H_{mi} , T_{jn} *and* b'_{mn} are previously defined by equations (74), (75), and (76). Equation(85) may be written in the matrix form

$$H\,A + A\,T + \alpha\,A = B' \tag{86}$$

which also has the same method of solution like equation (64).

6. Numerical results and discussion

Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions :

$$\begin{aligned} u(x,y) &= 0 & x &= -1 \\ u(x,y) &= 0 & x &= 1 \\ u(x,y) &= \tfrac{1}{2}(1-x^2) & y &= -1 \\ u(x,y) &= 0 & y &= 1 \end{aligned} \tag{87}$$

This problem has been considered by Doha [8] ,but by using doubly Chebychev polynomials This problem is symmetric about the y-axis, and its analytical solution is given by

$$u(x,y) = -\frac{2}{\pi^3}\sum_{i=1}^{\infty}\frac{(-1)^i}{(i-\frac{1}{2})^3}\frac{Cos(i-\frac{1}{2})\pi\ x\ Sinh(i-\frac{1}{2})\pi\ (1-y)}{Sinh(2i-1)\pi}$$

The approximate solution with M=N=16,is then

$$u(x,y) = \sum_{i=0}^{16}\sum_{j=0}^{16} a_{ij}C_i^{(\lambda)}(x)C_j^{(\lambda)}(y) \tag{88}$$

we write a program which automatically solves the problem (87) for any particular λ and evaluates the corresponding approximate coefficients. We shall take some special cases according to the values of the parameter λ

Case 1

We consider the case $\lambda = 0.0$, which means that the basis of expansion are Chebyshev polynomials of the first kind. The values of $\{a_{ij}\}$ for M=N=16 are given in Table 1. The results are in agreement with those obtained by Doha [8] . .

Case 2

We consider the case $\lambda = 0.5$, which means that the basis of expansion are Legendre polynomials .The values of $\{a_{ij}\}$ for M=N=16 are given in Table 2.

Case 3

We consider the case $\lambda = 1.0$,which means that the basis of expansion are Chebychev polynomials of the second kind .The values of $\{a_{ij}\}$ for M=N=16 are given in Table 3.

Case 4

We consider two cases for small and negative values of the parameter λ , one with $\lambda = -0.35$ and the second with $\lambda = -0.45$.The values of the set of coefficients $\{a_{ij}\}$ for M=N=16 for these two cases are given in Tables 4 and 5 respectively.

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	338236	-227157	76141	-21691	4418	-1028	267	-96	41	-20	10	-6	3	-2	1	-1	0
2	-173859	115620	-36958	9442	-1255	21	99	-6	31	-16	9	-5	3	-2	1	-1	0
4	4418	-1757	-1255	1373	-831	357	-124	33	-5	-3	3	-3	2	-1	1	0	0
6	267	-231	99	55	-124	118	-82	47	-23	10	-4	1	0	0	0	0	0
8	41	-39	31	-14	-5	18	-23	21	-16	10	-6	3	-2	1	0	0	0
10	10	-10	9	-7	3	1	-4	6	-6	5	-4	3	-2	1	-1	0	0
12	3	-3	3	-2	2	-1	0	1	-2	2	-2	2	-1	1	-1	0	0
14	1	-1	1	-1	1	-1	0	0	0	1	-1	1	-1	1	-1	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table.1 : Nonzero coefficints ($\times 10^6$) in the approximate solution of equation (87) with $\lambda=0$, M=N=16 .

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	96378	-143462	65224	-22042	4798	-1078	233	-70	26	-11	5	-3	1	-1	0	0	0
2	-101442	148674	-63735	18723	-2118	-412	479	-246	114	-53	26	-13	7	-4	2	-1	1
4	4798	-4524	-2118	3467	-2420	1075	-335	45	32	-38	27	-17	11	-6	4	-2	1
6	233	-316	479	-7	-335	404	-305	174	-79	26	-3	-5	6	-5	3	-2	1
8	26	-79	114	-97	32	38	-79	83	-66	43	-24	11	-4	1	1	-1	1
10	5	-16	26	-31	27	-14	-3	17	-24	24	-19	14	-9	5	-3	1	0
12	1	-4	7	-9	11	-9	6	-1	-4	7	-9	9	-7	5	-4	2	-1
14	0	-1	2	-3	4	-4	3	-2	1	1	-3	4	-4	3	-3	2	-1
16	0	0	1	-1	1	-1	1	-1	1	0	0	1	-1	1	-1	1	-1

Table.2 : Nonzero coefficints ($\times 10^6$) in the approximate solution of equation (87) with $\lambda=\frac{1}{2}$, M=N=16 .

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	99749	-135822	80569	-30084	6882	-1319	277	-66	19	-7	3	-1	1	0	0	0	0
2	-106931	163963	-79377	25215	-2428	-1097	987	-477	202	-84	36	-16	8	-4	2	-1	0
4	6882	-7113	-2428	5397	-4154	1891	-530	-8	128	-111	71	-41	22	-12	7	-3	2
6	277	-914	987	-217	-530	781	-632	364	-151	33	14	-25	22	-16	10	-6	3
8	19	-96	202	-231	128	33	-151	185	-154	10	-52	19	-2	-5	6	-5	4
10	3	-14	36	-61	71	-53	14	26	-52	57	-49	34	-20	1	-4	1	1
12	1	-3	8	-15	22	-26	22	-11	-2	14	-20	21	-18	14	-9	5	-3
14	0	-1	2	-4	7	-9	10	-9	6	-1	-4	8	-9	9	-8	6	-4
16	0	0	0	-1	2	-2	3	-3	4	-2	1	2	-3	4	-4	3	-3

Table . 3 : Nonzero coefficients($\times 10^6$) in the approximate solution of equation (87) with $\lambda = 1$, $M = N = 16$.

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	48838	-55660	8452	-1928	369	-89	27	-11	5	-2	1	-1	0	0	0	0	0
2	-49241	55932	-8198	1729	-255	34	-1	-2	1	-1	0	0	0	0	0	0	0
4	369	-232	-255	186	-97	40	-15	5	-2	1	0	0	0	0	0	0	0
6	27	-31	-1	13	-15	12	-8	4	-2	1	-1	0	0	0	0	0	0
8	5	-6	1	1	-2	2	-2	2	-1	1	-1	0	0	0	0	0	0
10	1	-2	0	0	0	0	-1	1	-1	0	0	0	0	0	0	0	0
12	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table . 4 : Nonzero coefficients($\times 10^6$) in the approximate solution of equation (87) with $\lambda = -0.35$, $M = N = 16$.

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	21338	-22437	1333	-276	52	-13	4	-2	1	0	0	0	0	0	0	0	0
2	-21394	22471	-1294	250	-38	6	-1	0	0	0	0	0	0	0	0	0	0
4	52	-29	-38	24	-12	5	-2	1	0	0	0	0	0	0	0	0	0
6	4	-4	-1	2	-2	1	-1	0	0	0	0	0	0	0	0	0	0
8	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table . 5 : Nonzero coefficients($\times 10^6$) in the approximate solution of equation (87) with $\lambda = -0.45$, $M = N = 16$.

From these Tables we deduce that the results corresponding to small and negative values of λ are superior to any of the others.
From this we conclude that the expansion based on Chebyshev polynomials of the first kind ($\lambda = 0.0$) is not always better than ultraspherical series.

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