WAVELET PROCESSING IN OPTICAL DIFFRACTION GRATING

معالجة الموجات لمحزز حيود ضوئى

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ملخص البحث :-

النمذجة الدقيقة للنبائط الفوتونية تعتبر أساسا لتطوير المكونات البصرية الجديدة ذات الأداء العالى المطلوبة حاليا ومستقبلا لأنظمة الإتصالات ذى النطاق العريض.هذا البحث يقدم تقنيات متنوعة لهذه النمذجة التى يستخدم العديد منها فى أدوات التصميم التجارية. يتضمن هذا البحث طريقة وافلت جاليركن وطريقة الفروق المحددة لإنتشار الشعاع الضوئى والتى طبقت على محزز حيود ضوئى حقيقى مصنع بتقنية التبادل الأيونى المزدوج. واوضحت النتائج أن طريقة وافلت جاليركن أفضل من طريقة الفروق المحددة فى شكل الخرج لشدة المجال الكهربى ولها القدرة على تمثيل الحل عند مستويات مختلفة وهذا يجعلها مفيدة جدا فى بعض المسائل الهندسية. لكن العمليات الحسابية لها معقدة وتأخذ وقتا كبيرا عن طريقة الفروق المحددة.

Abstract—Accurate modeling of photonic devices is essential for the development of new higher performance optical components required by current and future wide bandwidth communication systems. The goal of this paper is to formulate a Wavelet-Galerkin method to solve a paraxial wave equation numerically. The numerical simulation is applied to a real optical diffraction grating fabricated by a double--ion exchange technique.

The Wavelet solutions have much better precision but are slightly slower than the finite difference solution owing to the need to transform the samples from physical space into Wavelet space and back again. Although the Wavelet solutions require slightly more computational effort than the finite difference solution, the gains in accuracy, particularly with the higher order Wavelets, far outweigh the increase in cost. Furthermore, Wavelets have the capability of representing solutions at different levels of resolution, which makes them particularly useful for developing hierarchical solutions to engineer problems.

Key words —Wavelet-Galerkin method (WGM), finite-difference beam propagation method (FD-BPM), diffraction grating, Partial differential equation (PDE), methods of weighted residuals (MWR).

I. Introduction

In recent years, the subject of Wavelets has received much attention because of the comprehensive mathematical power and the good application potential of Wavelets in many interesting physical problems. The field of Wavelets, despite being relatively recent, is vast and is developing very rapidly. This is true in relation to theoretical aspects, as well as to applications of Wavelets [1].

The flexibility of Wavelet bases has already allowed their successful application to the resolution of various equations: Navier–Stokes equations (using divergence-free Wavelets) [2], Maxwell equations in time domain [3-5], and Schrödinger equations [6].

The development of the discretization formulations is based on weak form functionals and the use of Lagrange multiplier method to enforce the essential boundary conditions. Behiry [7] presented the application of the Wavelet-Galerkin method for one dimensional inhomogeneous diffusion equations subject to mass specification. Behiry and Hashish [8] presented a comparison of a Wavelet-Galerkin procedure with a Crank-Nicolson-Galerkin procedure for the diffusion equation subject to the specification of mass. The numerical results

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obtained show that the Wavelet-Galerkin procedure gives better results on the FD-BPM.

II. THEORY

II.A Galerkin Method

The Galerkin method is one of the best known methods for finding numerical solutions to partial differential equations (PDE). Its simplicity makes it perfect for many applications [9]. The Galerkin approach consists of finding a functional basis for the solution space of the equation, then projecting the solution on the functional basis, and minimizing the "residual" with respect to the functional basis. The translates of a Wavelet for all dilations form an unconditional orthonormal bases of $L^2(R)$ and the translates of a scaling function for all dilations form an unconditional orthonormal bases for $V_j \subset L^2(R)$, which is a great improvement over the standard polynomial basis or a trigonometric basis for the Galerkin method. Variable resolution levels inherent with Wavelets and multiresolution analysis allow for localizing small propagation distance scale variations of the solution. Furthermore, the properties of multiresolution analysis allow for fast switching of functional bases.

Consider the following boundary value problem:

$$L[u(x,y)] = 0, \text{ in } \Omega \subset R \tag{1}$$

where L is a linear differential operator and with boundary conditions on:

$$S(u) = 0, \text{ in } \partial\Omega \tag{2}$$

where $\partial\Omega$ is the boundary of the domain Ω and $S(u)$ is a function in u and/or its derivative. The Galerkin method assumes that u can be accurately represented by an approximate solution $\bar{u}_n$ given by:

$$\bar{u}_n = \sum_{i=1}^{n} a_i \varphi_i(x,y) \tag{3}$$

where the $\varphi_i(x,y)$ are known analytic functions, and the a_i are coefficients to be determined. Substituting (3) in (1) results in a nonzero residual $\bar{R}$ given by:

$$\bar{R}(a_1, a_2, \ldots, a_n, x, y) = \sum_{i=1}^{n} a_i L(\varphi_i) \tag{4}$$

In order to illustrate how the unknown coefficients a_i are uniquely determined, and hence obtain the approximate solution $\bar{u}_n$ the formation of the Galerkin system of equations is demonstrated. The Galerkin method is a member of a larger class of methods known as the methods of weighted residuals (MWR). The MWR is an optimizing criterion to select the specific numerical values for the parameters a_i to obtain the best solution. By "the best" it is meant that the solution is as close as possible, in some sense, to the exact solution. In these methods the coefficients a_i are determined by solving the system of equations:

$$(W_j, \bar{R}) = 0, \quad j = 1,2,3,\ldots,n \tag{5}$$

where the inner product of two arbitrary functions (f, g) on Ω is defined as:

$$(f, g) = \iint_{\Omega} f\, g\, dx dy \tag{6}$$

and from this relationship the method gets its name as $W_j(x,y)$ is referred to as a weight function or a test function. From the definition of the residual given in (4), the system of equations (5) can be written as:

$$\left(W_j, \sum_{i=1}^{n} a_i L(\varphi_i)\right) = 0, \quad j = 1,2,3,\ldots,n \tag{7}$$

or in matrix form as:

$$A\underline{a} = \underline{0} \tag{8}$$

where

$$A = \begin{bmatrix} (W_1, L(\varphi_1)) & (W_1, L(\varphi_2)) & \cdots & (W_1, L(\varphi_n)) \\ (W_2, L(\varphi_1)) & (W_2, L(\varphi_2)) & \cdots & (W_2, L(\varphi_n)) \\ \vdots & \vdots & \vdots & \vdots \\ (W_n, L(\varphi_1)) & (W_n, L(\varphi_2)) & \cdots & (W_n, L(\varphi_n)) \end{bmatrix} \tag{9}$$

and

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \underline{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

Since independent relationships are needed to obtain a unique solution for the unknown coefficients a_i's, it is clear that $W_j(x,y)$ must be independent functions.

If the W_j's are members of a complete set of functions, then as $n \to \infty$ the residual $\overline{R}$ must be orthogonal to every member of that complete set of functions. This implies that $\overline{R}$ converges to zero in the mean as $n \to \infty$. If $\overline{R}$ converges to zero in the mean and the boundary conditions are satisfied exactly, then it would be expected that the approximate solution to converge to the exact solution in the mean, that is:

$$\lim_{n\to\infty} \| u - \overline{u}_n \| = 0 \tag{11}$$

A variety of weighted residual methods is obtained depending on the choice of the weight function. Some frequently used weighted residual methods are the subdomain method, the point collocation method, the least square method, and the Galerkin method. The Galerkin method defines $W_j(x,y)$ as:

$$W_j(x,y) = \varphi_j, \quad 1 \le j \le n$$

It follows that in the Galerkin method the trial (test) functions are orthogonal to the residual and the matrix A in Eq.(9) are given by:

$$A = \begin{bmatrix} (\varphi_1, L(\varphi_1)) & (\varphi_1, L(\varphi_2)) & \cdots & (\varphi_1, L(\varphi_n)) \\ (\varphi_2, L(\varphi_1)) & (\varphi_2, L(\varphi_2)) & \cdots & (\varphi_2, L(\varphi_n)) \\ \vdots & \vdots & \vdots & \vdots \\ (\varphi_n, L(\varphi_1)) & (\varphi_n, L(\varphi_2)) & \cdots & (\varphi_n, L(\varphi_n)) \end{bmatrix} \tag{12}$$

II.B Multiresolution Analysis on $L^2(R)$

The multiresolution analysis presented by Mallat [10] is considered the main tool in merely all constructions of Wavelet bases. A multiresolution analysis is an increasing sequence $\{V_j\}_{j\in Z}$ of closed subspaces of $L^2(R)$ satisfying:

i. $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \ldots$

ii. $\bigcap_{j\in Z} V_j = \{0\}, \quad \bigcup_{j\in Z} V_j = L^2(R)$

iii. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
$f(x) \cup V_0 \Leftrightarrow f(x+1) \in V_0$

iv. $\exists \Phi(x) \in V_0$ such that $\{\Phi(x-k)\}_{k\in Z}$

form an orthonormal basis of V_0. As each subspace has a different resolution, hence the name multiresolution analysis. The sequence of subspaces V_j can be interpreted as follows. If P_j is the projection of f on V_j then P_j shows the details of f of size larger than 2^j. The higher j is the more details of f the projection shows. So, a function f can be considered as a limit of successive projections on these subspaces:

$$f = \lim_{j\to\infty} P_j \tag{13}$$

Define W_j to be the orthogonal complement of:

$$V_j \text{ in } V_{j+1}, \text{ i.e. } V_{j+1} = V_j \oplus W_j \tag{14}$$

This space is sometimes called the innovation space. The projection q_j of f *on* W_j tells what to be added to P_j to obtain details twice smaller, i.e.

$$P_{j+1} = P_j + q_j \tag{15}$$

Constructing a basis for V_j begins with $\Phi(j)$ known as the scaling function. Conditions *(iii)* and *(iv)* of the multiresolution analysis show that for an integer j the set of functions $\{\Phi_{j,k}(x)\}_{k\in Z}$ forms an orthonormal basis of V_j where:

$$\Phi_{j,k}(x) = 2^{j/2}\Phi\left(2^j x - k\right) \tag{16}$$

Since $V_0 \subset V_1$, there exists a sequence known as Wavelet filter coefficients, in the Hilbert sequences space L^2 such that:

$$\Phi(x) = \sum_k h_k \Phi(2x-k) \tag{17}$$

which is known as the scaling function dilation equation or the two-scale relation. Also a

function $\psi(x)$, known as mother Wavelet function, can be defined such that $\{\psi(x-k)\}_{k\in Z}$ forms an orthonormal basis of W_0 and $\{\psi(x-k)\}_{k\in Z}$ forms an orthonormal basis of W_j where:

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \tag{18}$$

Similarly $\psi(x)$ satisfies its own dilation equation:

$$\psi(x) = \sum_k g_k \psi(2x-k) \tag{19}$$

Thus two possible orthonormal bases of V_j are $\{\Phi_{j,k}(x)\}_{k\in Z}$, and:

$$\{\Phi(x-k)\cup\psi_{j,k}(x) \mid i = 0,1,\ldots,j-1\}_{k\in Z}$$

It is noticed from multiresolution definition that W_j's are mutually orthogonal and that:

$$V_{j+1} = V_j \oplus W_j = V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_j \tag{20}$$

and

$$L^2(R) = \bigoplus_{j\in Z} W_j \tag{21}$$

From the above analysis, a definition to Wavelets can be given.

Definition (Wavelet) A Wavelet is a function $\Psi(x)\in L^2(R)$ such that the family of functions:

$$\Psi(x)_{j,k} = 2^{j/2}\Psi\left(2^j x - k\right) \tag{22}$$

where j and k are arbitrary integers, $\psi(x)_{j,k}$ is an orthonormal basis of the Hilbert space $L^2(R)$. Mallat [10] presented an algorithm, known as Mallat algorithm, to change from a certain representation scale to another.

III. Solving Scalar Wave Equation

As shown in the previous section, applying properties of multiresolution analysis [10-11] to the Galerkin method offers a few improvements over traditional trial functions. However, it restricts our solutions to elements of V_j. In solving partial differential equations arising from physical phenomena solutions will belong to the class of finite energy surfaces, or $L^2(R^2)$. Thus it would be more beneficial to have a scheme that will solve the PDE in $L^2(R^2)$ rather than in $V_j \subset L^2(R)$. This is where the Wavelets role is, since they can form orthogonal bases of $L^2(R^n)$.

Let us see how a hierarchy of Wavelet solutions to PDEs may be developed using scaling function bases. In order to demonstrate the Wavelet technique, we consider the three dimensional second order differential equation based on Maxwell's equations [12] and the scalar wave equation for the propagating beam problem is deduced as [13]:

$$\frac{\partial^2 u}{\partial^2 z} - 2i_m\beta\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(K^2 - \beta^2\right)u = 0 \tag{23}$$

where $K^2 = k_0^2 n^2$, $\beta^2 = k_0^2 n_0^2$, $i_m = \sqrt{-1}$ and k is known as the wave-number. In free space, $k_0 = 2\pi/\lambda$, n is the refractive index distribution of waveguide structure, and n_0 the reference refractive index to be appropriately chosen.

At this point, the above equation is completely equivalent to the exact Helmholtz equation [14]. Except that it is expressed in terms of u. It is now assumed that the variation of u with z is sufficiently slow so that the first term of u above can be neglected with respect to the second; this is the familiar slowly varying envelope approximation, and in this context it is also referred to as the paraxial or parabolic approximation. With this assumption and after slight rearrangement, the Eqn. (23) reduces to:

$$\frac{\partial u}{\partial z} = \frac{1}{2i_m\beta}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(K^2 - \beta^2\right)\right]u \tag{24}$$

This is the basic beam propagation equation in three dimensions (3D). Simplification to two dimensions (2D) is obtained by omitting any dependence on y. Given an input field $u(x,y,z) = 0$, the above equation determines the evolution of the field in the space $z > 0$. It is important to recognize what has been gained and lost in the above approach. First, the factoring of the rapid phase variation allows the slowly varying field to be represented numerically on a longitudinal grid (i.e., along z) that can be much coarser than the wavelength

for many problems, contributing in part to the efficiency of the technique. Second, the elimination of the second derivative term in z reduces the problem from a second-order boundary value problem requiring iteration or eigenvalue analysis, to a first-order initial value problem that can be solved by simple "integration" of the above equation along the propagation direction. This latter point is also a major factor in determining the efficiency of the algorithm, implying a time reduction by a factor of at least of the order of (the number of longitudinal grid points) compared to full numerical solution of the Helmholtz equation. The above benefits have not come without a price. The slowly varying envelope approximation limits consideration to fields that propagate primarily along the axis (i.e., paraxiality) and also places restrictions on the index contrast (more precisely, the rate of change of index with z, which is a combination of index contrast and propagation angle). In addition, fields that have a complicated superposition of phase variation, such those existing in multimode devices such as multimode-interference may not be accurately modeled if the phase variation is critical to device behavior. A second key issue beyond the above restrictions on the variation of u is that the elimination of the second derivative also eliminates the possibility for backward traveling wave solutions; thus devices for which reflection is significant will not be accurately modeled. In the following section the numerical solution of Eqn. (24) discussed above is considered.

III. A Numerical Solution Using Finite Difference (FD-BPM)

In the finite-difference approach, the field in the transverse (x,y) plane is represented only at discrete points on a grid, and at discrete planes along the longitudinal or propagation direction z [34]. Given the discretized field at one plane, the goal is to derive numerical equations that determine the field at the next plane. This elementary propagation step is then repeated to determine the field throughout the structure. For simplicity, the approach is illustrated for a scalar field in 2D (x,z). Let u_i^n denote the field at transverse grid point i and longitudinal plane n, and assume that the grid points and planes are equally spaced by Δx and Δz apart, respectively. In the Crank–Nicholson method Eqn. (24) is represented at the mid plane between the known plane and the unknown plane as follows:

$$\frac{\partial u}{\partial z} = \frac{1}{2i_m\beta}\left[\frac{\partial^2 u}{\partial x^2} + \left(K^2 - \beta^2\right)\right]u \tag{25}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta z} = \frac{1}{2i_m\beta} \left[\frac{\delta^2}{\Delta x^2} + \left(K\left(x_i - z_{n+1/2}\right)^2 - \beta^2\right)\right]\frac{u_i^{n+1} + u_i^n}{2} \tag{26}$$

Here δ^2 represents the standard second order difference operator $\delta^2 u_i = u_{i-1} - 2u_i + u_{i+1}$ and $z_{n+1/2} = z_n + \Delta z/2$. The above equation can be rearranged into the form of a standard tridiagonal matrix equation [15] for the unknown field u_i^{n+1} in terms of known quantities, resulting in

$$-au_{i-1}^{n+1} + bu_i^{n+1} - au_{i+1}^{n+1} = au_i^n + cu_i^n + bu_i^n \tag{27}$$

where

$$a = \frac{\Delta z}{2\Delta x^2}$$

$$b = \frac{\Delta z}{\Delta x^2} - \frac{\Delta z\, k_0^2}{2}\left(n_i^2\left(z + \Delta z\right) - n_0^2\right) + 2jk_0n_0$$

$$c = \frac{\Delta z}{\Delta x^2} - \frac{\Delta z\, k_0^2}{2}\left(n_i^2\left(z\right) - n_0^2\right) + 2jk_0n_0$$

These results in a tridiagonal system of linear equations, which can be solved very efficiently. The solution to this system of equations can be also shown to be stable.

The numerical simulations are applied to planar diffraction grating waveguide [16] as shown in Fig. (1). the measured values of the grating of the effective refractive index of the guided wave mode (n_{e1} and n_{e2}) have done with a He-Ne laser (λ=0.6328μm) through prism couplers are equal to n_{e1} = 1.512689, n_{e2} = 1.513739 the periodicity of the grating is ($\wedge$ = 10 μm) and the incident angle of the

optical wavelength as a Bragg angle which is given by:

$$\theta_B = \sin^{-1}\left[\frac{\lambda}{2n_{el}\Lambda}\right] \quad \theta_B(\lambda_1) = 1.1985085^{\circ}.$$

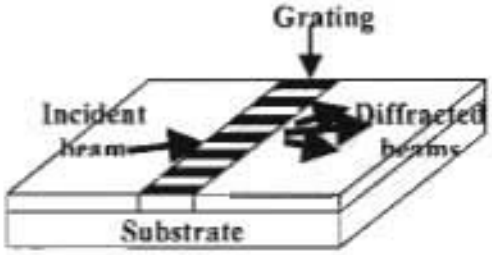


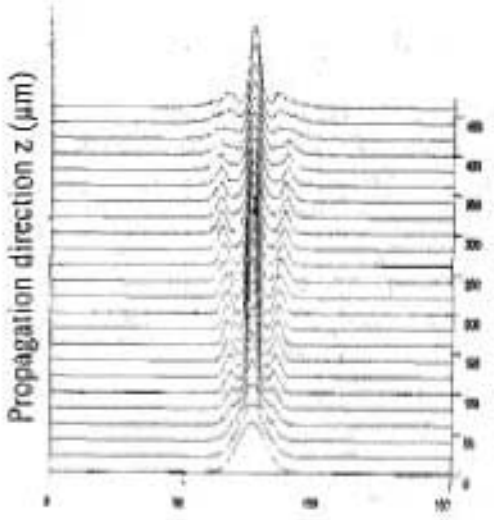
Fig. (1) Phase diffraction grating

A Gaussian profile whose full width at half maximum is w_o=10 μm is used as the initial field profile $U_O(x_i, z_o) = exp\left(-x_i^2 / w_o^2\right)$. The computational window is 150μm for all the simulations. For all methods the accuracy of the results depend on the number of grid points N in the transverse, i.e., x direction, and the size of the propagation steps, Δz in the direction of propagation, i.e., in *z-direction*. In all methods N=128, Δz=1μm and the propagation interaction length is 474μm.

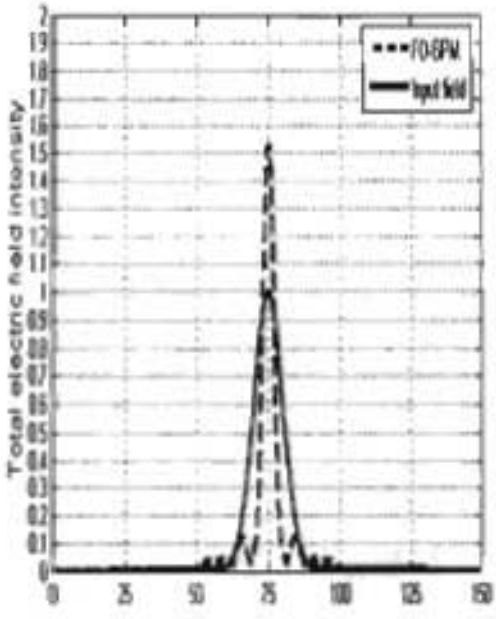
The simulation results are shown in Fig. (2-a,b) for a beam intensity at interaction length 474μm. For more details refer to [17] for the same authors.

III.B Numerical Solution Using Wavelet-Galerkin Method (WGM)

The Wavelets method offers several advantages over the traditional methods such as (1) the MRA ability of Wavelets provides a local means of developing a hierarchy of solutions; (2) the solution of the Wavelet method can be refined in regions of high gradient without having to regenerate the mesh for the entire problem and (3) the trade-off between continuity and compact support is well balanced. Full details of the Wavelet method for 1D PDEs are referred to [18] and for 2D PDEs to [19-20]. For those applications periodic Wavelets are used to cooperate boundary conditions.



(a) Transverse direction x (μm)



(b) Transverse direction x (μm)

Fig. (2) A grating BPM solution using FD
(a) Field profile through 474μm of the grating.
(b) Output beam intensity of 474μm of the grating.

But only few applications of Wavelet-Galerkin method have been reported in electromagnetics.

This section investigates the feasibility of Wavelet-Galerkin method for computating of electromagnetic fields inside optical planar diffraction grating.

III.B.1 Connection coefficients

Since the scaling function and Wavelets do not have an explicit analytical expression but are implicitly determined by the Wavelet filter coefficients, it is necessary to develop algorithms to compute several integrals, called the connection coefficients. These coefficients, involving the scaling function and/or its derivatives, occur in the application of the Wavelet-Galerkin procedure to differential and integral equations. Consider Daubechies scaling function satisfying the dilation Eqn. (17) with $k=0, 1,\ldots,N-1$, then the following are the connection coefficients used in the following sections:

$$M_k^m(x)=\int_0^x y^m \Phi(y-k)\,dy \tag{28}$$

$$\Gamma_k^n(x)=\int_0^x \Phi^{(n)}(y-k)\,\Phi(y)\,dy \tag{29}$$

$$A_k^{m,n}=\int_0^x y^m \Phi^{(n)}(y-k)\,\Phi(y)\,dy \tag{30}$$

and

$$\Phi^{(n)}=\frac{\partial^n \Phi(x)}{\partial x^n} \tag{31}$$

The applications of Wavelet-Galerkin method were at first limited to the cases where the problem domain is unbounded or the boundary conditions are periodic [20]. Algorithms for computing these connection coefficients at different values of x on bounded intervals given in [21] motivated applying the method to bounded domains [22-26].

III.B.2 Wavelet-Galerkin Method (WGM)

In this section the WGM is applied to Eqn. (24) and consider the initial condition

$$u(x,y,0)=v(x,y),\ 0<x<L_x,\ 0<y<L_y \tag{32}$$

where L_x and L_y are real constant and z denotes the propagation direction of light. Let the solution $u(x,y,z)$ of (24) be approximated by the j^{th} level Wavelet series.

$$\bar{u}_j(x,y,z)=\sum_{k=N-2}^{2^j L_x-1}\sum_{\ell=N-2}^{2^j L_y-1} u_{j,k,\ell}(z)\,\Phi_{j,k,\ell}(x,y),\ j>0 \tag{33}$$

Substituting (33) in (24) yields:

$$\sum_k\sum_\ell u_{j,k,\ell}(z)\,\Phi_{j,k,\ell}(x,y)=\frac{1}{2i_m\beta}\left(\begin{array}{l}\sum_k\sum_\ell u_{j,k,\ell}(z)\,\Phi^{*}_{j,k}(x)\,\Phi_{j,\ell}(y)+\\ \sum_k\sum_\ell u_{j,k,\ell}(z)\,\Phi^{*}_{j,\ell}(y)+\left(k^2-\beta^2\right)\\ \sum_k\sum_\ell u_{j,k,\ell}(z)\,\Phi_{j,k}(x)\,\Phi_{j,\ell}(y)\end{array}\right) \tag{34}$$

where

$u_{j,k,\ell}(z)=\frac{\partial u_{j,k,\ell}(z)}{\partial z}$, $\dot{\Phi}_{j,k}(x)=\frac{\partial^2 \Phi_{j,k}(x,y)}{\partial x^2}$, and $\dot{\Phi}_{j,\ell}(x)=\frac{\partial^2 \Phi_{j,k,\ell}(x,y)}{\partial y^2}$ For ℓ and S having the same limits as k and i, respectively, the Galerkin discretization scheme yields:

$$\sum_k\sum_\ell \dot{u}_{j,k,\ell}(z)\,a_{x,\ell,k}\,a_{y,S,i}=\frac{1}{2i_m\beta}2^{2j}\left(\sum_k\sum_\ell u_{j,k,\ell}(z)\,c_{x,\ell,k}\,a_{y,S,i}+\sum_k\sum_\ell u_{j,k,\ell}(z)\,a_{x,\ell,k}\,c_{y,S,i}+\left(k^2-\beta^2\right)\sum_k\sum_\ell u_{j,k,\ell}(z)\,a_{x,\ell,k}\,a_{y,S,i}\right) \tag{35}$$

where

$$a_{x,\ell,k}=2^j\int_0^{L_x}\Phi\left(2^j x-k\right)\Phi\left(2^j x-\ell\right)dx=\Gamma^0_{k-\ell}\left(2^j L_x-k\right)-\Gamma^0_{k-\ell}\left(2^j(0)-\ell\right) \tag{36}$$

$$a_{y,S,i}=2^j\int_0^{L_y}\Phi\left(2^j y-k\right)\Phi\left(2^j y-S\right)dy=\Gamma^0_{i-S}\left(2^j L_y-S\right)-\Gamma^0_{i-S}\left(2^j(0)-S\right) \tag{37}$$

$$c_{x,\ell,k}=2^j\int_0^{L_x}\Phi^{*}\left(2^j x-k\right)\Phi\left(2^j x-\ell\right)dx=\Gamma^2_{k-\ell}\left(2^j L_x-k\right)-\Gamma^2_{i-S}\left(2^j(0)-\ell\right) \tag{38}$$

$$c_{y,S,i} = 2^j \int_0^L \Phi^*(2^j y - k)\Phi(2^j y - S)dy = \Gamma^2_{i-S}(2^j L_y - S) - \Gamma^2_{i-S}(2^j(0) - S) \quad (39)$$

define the matrices

$$A_x = [a_{x,j,k}], \quad A_y = [a_{y,S,j}], \quad C_x = 2^{2j}[c_{x,j,k}]$$

$$C_y = 2^{2j}[c_{y,S,i}], U = [u_{j,i,k}], \quad \dot{U} = [u_{j,i,k}].$$

Then (35) may be written as

$$(A_y U C_x^t + C_y U A_x^t) = A_y U A_x^t, \quad (40)$$

where A_x^t and C_x^t are the transpose of the matrices A_x and C_x respectively. Using the Kronecker product the system takes the form

$$\bar{C}\,\bar{U} = \bar{A}\,\dot{\bar{U}}, \quad (41)$$

where $\bar{C} = C_x \otimes A_y + A_x \otimes C_y$

$$\bar{A} = A_x \otimes A_y, \bar{U} = vec(U)\dot{\bar{U}} = vec(\dot{U}).$$

Using Crank-Nicolson scheme to approximate the propagation in z direction, (41) becomes

$$\left(\bar{A} - \frac{dz}{2}\bar{C}\right)\bar{U}^{n+1} = \left(\bar{A} - \frac{dz}{2}\bar{C}\right)\bar{U}^n, \quad (42)$$

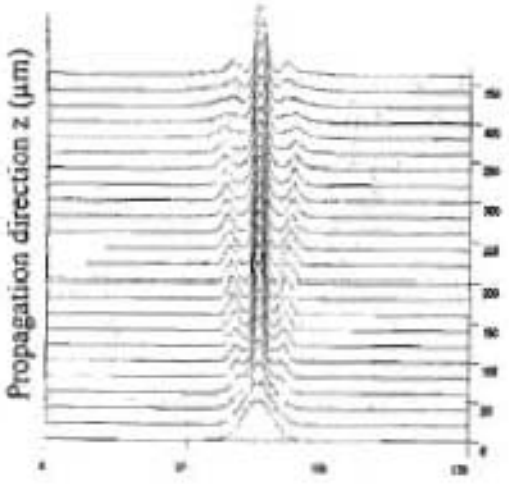
Eqn. (42) describes an iterative scheme to compute the value of $u(x,y,z_{n+1})$ from $u(x,y,z_n)$ where $z_n = ndz$. The next step is to obtain the initial value of the solution $u(x,y,0)$ in vector form and then enforce the boundary conditions to the system. Substituting (33) in (32) and applying the Galerkin discretization yields

$$\bar{A}\bar{U}^* = vec\left(\left[\int_0^{Ly}\int_0^{Lx} v(x,y)\Phi_{n,\ell,S}(x,y)\,dxdy\right]\right) \quad (43)$$

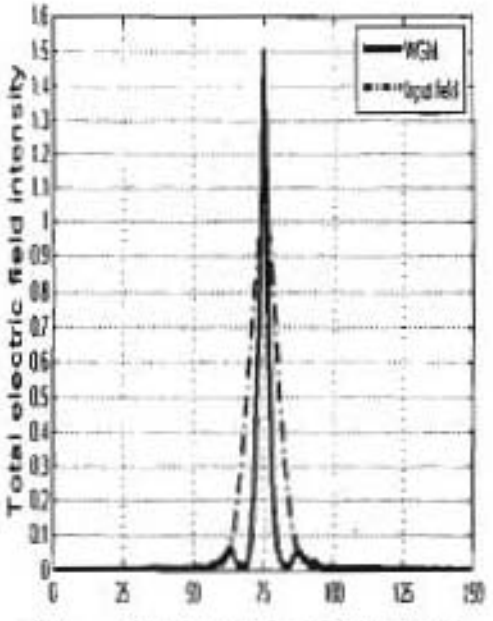
The simulation results are shown in Fig. (3-a,b) for a beam intensity at interaction length 474μm.

IV. Comparison of results

The Wavelet solutions have much better precision [26] but are slightly slower than the finite difference solution owing to the need to transform the samples from physical space into Wavelet space and back again. This overhead becomes less significant as the sample size increases.



(a) Transverse direction x (μm)



(b) Transverse direction x (μm)

Fig. (3) A grating solution using WGM

(a) Field profile through 474 μm of the grating
(b) Output beam intensity of 474 μm of the grating at approximation level j=3.

More importantly, there is a negligible variation in computation time as the support of the Wavelet increases. Thus the D6 Wavelet solution compares extremely favorably with the finite difference solution as shown in Fig. (4).

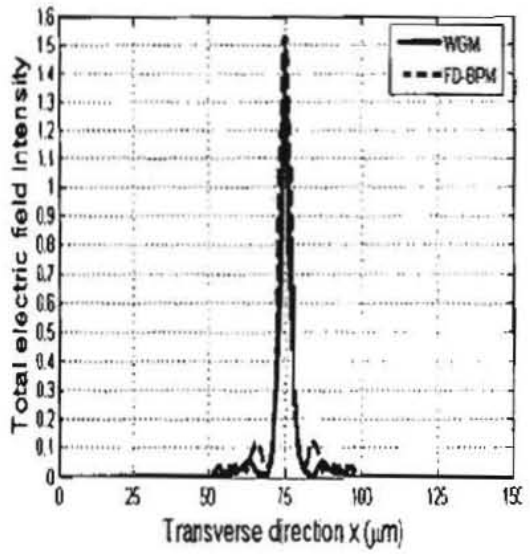


Fig.(4) Output beam intensity of 474 μm of the grating using WGM at approximation level *j=3* and FD-BPM.

V. Conclusions

In engineering problems, we often require a quick rough estimate of the solution at the preliminary stage, which may later be refined as the design or investigation progresses. Wavelets have the capability of providing a multilevel description of the solution . The multiresolution property of Wavelets, along with their localization property, suggests that we may obtain an initial coarse description of the solution with little computational effort and then successively refine the solution in regions of interest with a minimum of extra effort. Preliminary research indicates that Wavelets are a strong contender to finite elements in this respect, however, further research is still required on the subject.

The Wavelet method has been shown to be a powerful numerical tool for the fast and accurate solution of the scalar wave equation. Solutions obtained using the Daubechies coefficient Wavelets have been compared with the finite difference solution and the Wavelet solutions have been found to converge much faster than the finite difference solution [26]. Although the Wavelet solutions require slightly more computational effort than the finite difference solution, the gains in accuracy, particularly with the higher order Wavelets, far outweigh the increase in cost. Furthermore, Wavelets have the capability of representing solutions at different levels of resolution, which makes them particularly useful for developing hierarchical solutions to engineer problems.

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