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Evaluation of Different Discretization Schemes for Non-Orthogonal Grids and Highly Skewed Flow

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ABSTRACT
The current paper presents comparisons between four different discretization schemes for non-orthogonal grids in the skewed cavity and highly skewed flow in the curved cavity. These schemes are upwind differencing scheme (UDS), upwind differencing scheme with numerical diffusion (UDS-ND), central differencing scheme (CDS), and Quadratic upwind interpolation for convective kinematics (QUICK). The comparison between the selected schemes for highly skewed flow in curved cavity indicated that the upwind differencing scheme with numerical diffusion is the best choice in terms of accuracy and computational cost. In addition, the comparisons between the present results and previous results from the literature indicate that the current procedure which is more suitable for general purpose codes can produce computational results which are in close agreement with those obtained from body fitted and polar coordinates systems. For the non-orthogonal grids in the skewed cavity, all the tested schemes produce close results when they are compared with the benchmark solution. However, the UDS-ND requires fewer number of iteration and shorter computational time. Despite the upwind differencing scheme with numerical diffusion being the first-order scheme, its accuracy is very close to the second and third-order schemes. Therefore, the UDS-ND is recommended for general-purpose code because its stability is higher than the higher-order scheme and the computational time is lower.
Keywords: *Skewed Flow, Numerical Diffusion, Non-orthogonal, Curved Cavity, Discretization.*

1. Introduction

One of the greatest problems in numerical solution is the truncation error. This error appears clearly when transforming the non-linear equations system to an algebraic system of equations on the grids as mentioned by Lilek and Perić [1]. The most cell center properties (velocity, pressure, density……etc.) of the control volume are the unknowns in the algebraic equation system. The cell face properties should be computed based on interpolation or extrapolation of the neighbor cell center properties. Due to the application of series transformation to calculate the cell face properties, some terms in this series should be neglected.

The discretization schemes for a convective term in Navier-Stokes equations aim to reduce the truncation errors in order to enhance the performance of the numerical scheme. Finding the most accurate scheme is one of the challenges in this field. Many researches introduce discretization schemes such as, upwind differencing schemes, which are widely used although it has a first-order truncation error.

Many updates for the upwind difference schemes had been made such as, second-order upwind (SOU) which extrapolates the cell face property with two upstream cell center properties, see for example, Shyy, et al. [2] and Zurigat and Ghajar [3]. Also, there is another technique to have the higher-order truncation error by including the second term in the Taylor series expansion and treating it as a diffusion part as presented in Ferziger and Perić [4].

When the velocity direction and the grid orientation have a large angle between them (they aren't aligned), the accuracy of the upwind differencing scheme is reduced due to the skewness of the flow. Raithby [5] introduced a modification on the upwind differencing scheme known as the skew-upwind difference scheme (SUDS) to reduce this error. SUDS evaluate the cell face center property based on two upstream neighbors' cell center properties, as shown in Fig. 1. This scheme needs more than five nodes in the numerical study, it extends to have other four nodes as, SW, NW, NE, and SE. One of its drawbacks is the higher computation process needed for each cell face center. There are six different possibilities for interpolation for each cell face property as referred by Huang, et al. [6]. Busnaina, et al. [7] modified the (SUDS) to the skew upwind weighted differencing scheme (SUWDS) to reduce the oscillation and the numerical diffusion.

One of the third-order truncation error schemes used is the quadratic upwind differencing scheme, which gives an interpolation by three neighbors' cell center nodes. According to the flow direction, one downstream and two upstream nodes should be used to approximate the function of the property as a second-order curve with the distance. Many researchers studied the performance and stability of these discretization schemes as Leonard [8] and Johnson and Mackinnon [8]. Until now there are many other discretization schemes discussed by many authors, one of the oldest schemes is the central difference scheme (CDS) which gives a linear interpolation between the two neighbor cells as introduced by Patankar [9].

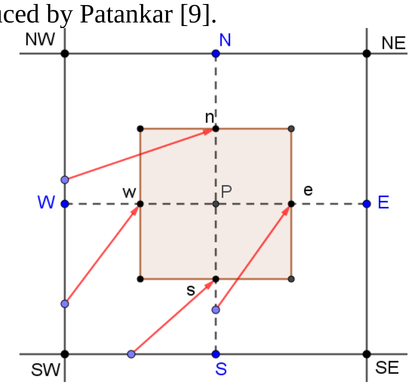


Fig. 1. Skew upwind differencing scheme.

All these schemes among other schemes had been used broadly and many investigators compared them in different geometrical test cases. Because of the great variety in these schemes, finding the better discretization scheme (accuracy, computational time, and stability) is one of the most challenging in this research area.

The lid-driven cavity was used by many authors (such as Schreiber and Keller [10], Huang, et al. [6], Runchal [11], Shyy, et al. [2], Biagioli [12], Yu, et al. [13], Jin and Tao [14], [15]) to validate the discretization schemes. The most geometrical shapes used are square cavity with a benchmark solution introduced by Ghia, et al. [16] and Botellaab and Peyretab [17].

Biagioli [12] compared central upwind (CU), second-order upwind (SOU) and total variation diminishing schemes (TVD). He concluded that (TVD) reach a grid-independent solution ten times faster than (CU) while (SOU) was seven times faster than (CU). Huang, et al. [6] discussed four different discretization schemes (SUDS, QUICK, PLDS, and LOADS) for three different test cases. They found that the QUICK scheme with coarse-grids (22×22) performs better than other schemes with finer grids (42×42).

Shyy et al. [2] introduced a comparison between (CDS) and three different schemes based on the (SOU). A, B, and C are the names of these three schemes, scheme A was based on an approximation of two upstream neighbors' nodes by finite difference technique. While scheme B used a finite volume approximation. The last one, scheme C, was derived by integrating the fluxes over the control volumes. They found that scheme C was better than the other two schemes.

The Study of the instability and boundedness of the numerical scheme based on the normalized properties for different discretization schemes was introduced by Yu et al. [13]. They introduced three new schemes based on stability analysis. After comparing all discretization schemes in simple test cases, they concluded that the new three schemes give better results than the other schemes. Also, Jin and Tao [14] introduced an instability study for 14 hybrid schemes based on the second-order schemes. The difference between these schemes is the coefficient of the pole node which varied from -2 to 10. They concluded that with increasing this coefficient the solution has better stability, but the false diffusion increases for higher *Re* Number. They found that the more the scheme is accurate the more the scheme stability is reduced. For the test case studied they recommend the coefficient should be in the range from 0.5 to 2. Jin and Tao [15] extended their study for 17 hybrid schemes based on the third-order differencing schemes which give better stability and good balance between accuracy and false diffusion as compared with their previous study [17].

Study of the accuracy and stability of the interpolation and extrapolation of the cell face properties introduced by all previous studies in an orthogonal grid, for more generalization of the convection schemes the non-orthogonal grids are very challenging and widely used for different engineering geometries. Demirdžić, et al. [18] introduced a bench-mark solution for a skewed cavity with two different angles. Oosterlee, et al. [19] also gave a bench-mark solution for two non-orthogonal grids as a skewed cavity and L-shaped cavity. Based on the work of Demirdžić, et al. [18], Erturk and Dursun [20] presented more data as a benchmark solution for more skewed angles.

Many authors compared the benchmark test case of the skewed cavity with different numerical techniques see, for example, Roychowdhury, et al. [21], Tucker and Pan [22], Li, et al. [23], Choi, et al. [24], Dalal, et al. [25], Paramane and Sharma [26], Lehnhäuser and Schäfer [27], Murali and Rajagopalan [28], Cheng, et al. [29], Kooshkbaghi and Lessani [30] and Kumar, et al. [31] and many others. Some of these studies were interested in the performance of different schemes. For an instant, Roychowdhury, et al. [21] applied for a first-order upwind and quadratic upwind for two different angles and with grid sizes from 41×41 to 129×129 and concluded that the QUICK scheme is better for higher non-orthogonal grids. Tucker and Pan [22] used orthogonal grids and treated the boundary walls as a cutting in the cell with special integration from the governing equations. Li, et al. [23] used the modified TVD scheme for the convictive flux with a triangle grid. While Dalal, et al. [25] used unstructured grids and used the weighted average of the central difference and first-order upwind for the convective flux.

Choi, et al. [24] compared the QUICK and hybrid schemes with four different discretization schemes and deduced that the QUICK gives better accuracy. QUICK, SOU, CDS, and FOU had been compared in a square and skewed lid-driven cavity by Paramane and Sharma [26]. In these test cases, the first order upwind is the worst scheme in the accuracy compared with the other three differencing schemes. The study used two grid sizes as (20×20) and (80×80. All of the comparisons between these schemes (for the best of our knowledge) didn't compare the first-order upwind with numerical diffusion.

One of the challenging problems in numerical solution is the skewness of the flow when there is a difference between the gridlines and the velocity components. This problem appears clearly when introducing a generalized numerical code for complex geometries. The curved cavity is a very challenging test case used to validate the discretization scheme for this purpose. Many researchers investigate the curved cavity with a different formulation of the Navier-Stokes equation as a Cartesian coordinate, polar coordinate, and stream function equations. Fuchs and Tillmark [32] introduced experimental and numerical study for the polar cavity for two Reynolds numbers of 60 and 350. A benchmark solution for a polar-driven cavity for a wide range of Reynolds number (Re≤17500) was introduced by Erturk [33]. Many authors validated their numerical code with the experimental data presented by Fuchs and Tillmark [32]. See, for example, Rosenfeld, et al. [34], Zang and Street [35], Chang and Cheng [36], Lei, et al. [37], Qu, et al. [38], Darbandi and Vakilipour [39], Kooshkbaghi and Lessani [30], Yu and Tian [40], Sen and Kalita [41], Kozyrakis, et al. [42] and Erturk and Gokcol [43].

Most studies of the polar cavity use the polar coordinates and stream function-vorticity equations. The others use the Cartesian coordinates with the coordinate transformation technique. These techniques are not suitable for complex geometries. In addition, the evaluation of discretization schemes for convective fluxes in polar cavities had not been studied for our best knowledge.

In the current paper, four different discretization schemes (UDS, UDS-ND, QUICK, CDS) will be evaluated for two test cases. The first test case is the highly skewed flow to grids in the curved cavity and the second one is non-orthogonal grids in a skewed cavity.

2. Mathematical Modeling

2.1. Governing Equations

The two-dimensional Navier-Stokes equations and continuity equation in Cartesian coordinates can be used to describe the flow in our test cases. Assume the flow is laminar, incompressible, and steady.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\rho\left[\frac{\partial(uu)}{\partial x}+\frac{\partial(uv)}{\partial x}\right]=-\frac{\partial p}{\partial x}+\mu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)+\left(\frac{\partial^2 u}{\partial y^2}\right)\right], \tag{2}$$

$$\rho\left[\frac{\partial(uv)}{\partial x}+\frac{\partial(vv)}{\partial x}\right]=-\frac{\partial p}{\partial y}+\mu\left[\left(\frac{\partial^2 v}{\partial x^2}\right)+\left(\frac{\partial^2 v}{\partial y^2}\right)\right], \tag{3}$$

2.2. Discretization schemes

The general form of conservation equation can be written in integral form as given by Ferziger and Peric [4] as:

$$\iint_S \rho\varphi V\cdot ndS=\iint_S \mu\nabla\varphi\cdot ndS+\iiint_{Vol.} q_\varphi dVol. \tag{4}$$

Where φ is the generalized dependent variable, μ is the dynamic viscosity, q_φ is the source term, and ***n*** is the unit normal vector outward from the center of the cell face as shown in Fig. 2.

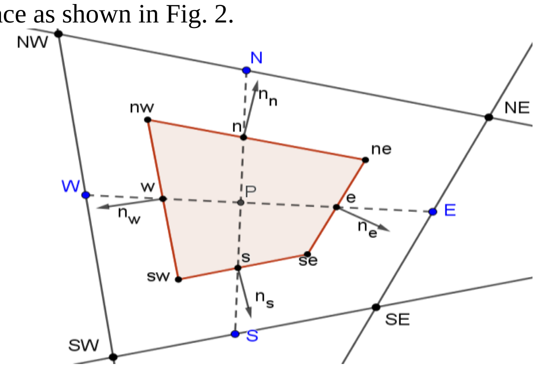


Fig. 2. Two-dimensional control volume

The conservation equation can be dived into three terms as convection, diffusion, and source term. Our present study will focus on the convection term which can be approximate as:

$$\iint_S \rho\varphi V\cdot ndS = \sum_f \rho(V\cdot n\Delta S)\varphi_f = \sum_f (\dot m_f\cdot\varphi_f). \quad (5)$$

Where $f = e, w, n, s$ the cell face center's properties.

All physical properties are known at the cell centers (P, E, W, N, S) while at the cell face center they are unknowns. Many discretization schemes can be used to calculate the cell face properties, as discussed in the previous section. A comparison between four different schemes (UDS, UDS-ND, QUICK, and CDS) will be introduced.

2.2.1. Upwind Difference Schemes (UDS)

The cell face is approximated by the first neighbor cell center property based on the flow direction which can be estimated from the Taylor series expansion as the flow direction from p to E, i.e., $(\dot m_e > 0)$. Thus, the expansion of Taylor series about p gives:

$$\varphi_e = \varphi_p + (x_e - x_p)\frac{\partial\varphi}{\partial x_p} + (y_e - y_p)\frac{\partial\varphi}{\partial y_p} + H \quad (6)$$

The UDS discretization scheme is a first-order truncation error (the first partial derivatives had been neglected) thus it gives ($\varphi_e = \varphi_p$) as shown in Fig. 3.

In general, ($\varphi_f = \varphi_U$).

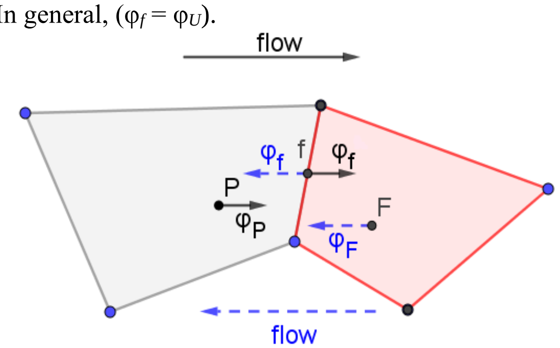


Fig. 3. Upwind differencing scheme

2.2.2. Upwind Difference Schemes with numerical diffusion (UDS-ND)

This method is based on (UDS) put with second-order truncation error (the second partial derivatives had been neglected) so that the cell face property can be calculated by:

$$\varphi_e = \varphi_p + (x_e - x_p)\frac{\partial\varphi}{\partial x_p} + (y_e - y_p)\frac{\partial\varphi}{\partial y_p} \quad (7)$$

In general

$$\varphi_f = \varphi_U + (r_f - r_U)\cdot\nabla\varphi_U \quad (8)$$

So, the first term is treated as a convective term while the second one takes the form of the diffusion term.

2.2.3. Linear Interpolation (central difference scheme) (CDS)

The linear interpolation is widely used to evaluate the cell face property from the two-neighbor cell center property. This method also has a second-order truncation error (the second partial derivatives had been neglected). The difference between this scheme and the previous one is that the first partial derivative is vanished in CDS by solving two equations of Taylor expansion at upstream and downstream control volume center property.

Usera, et al. [44] introduced a linear interpolation for central difference as shown in Fig. 4

$$\varphi_f = \varphi_F * \alpha_{PF} + \varphi_P * (1 - \alpha_{PF}) + (\nabla\varphi)_f \cdot \overline{f'f} \quad (9)$$

Where, $f = e, w, s, n$, and $F = E, W, S, N$

$$\alpha_{PF} = \frac{|\overline{Pf}|}{|\overline{Pf}| + |\overline{fF}|} \quad (10)$$

$$\overline{fF} = \alpha_{PF} * \overline{Ff} + (1 - \alpha_{PF}) * \overline{Pf} \quad (11)$$

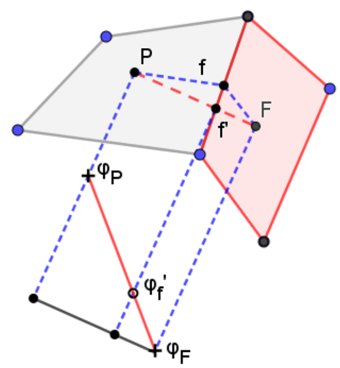


Fig. 4. Central difference scheme.

2.2.4. Quadratic upwind interpolation for convective kinematics (QUICK):

The QUICK discretization scheme is a third-order truncation error approximation based on the Taylor series expansion by neglecting the third partial derivatives. Leonard [45] presented a formulation of a QUICK differencing scheme for interpolation in the rectangular control volume. Three different cell center properties are needed as (U, UU, D) upstream, up-upstream, and downstream as shown in Fig. 5.

But for the generalized coordinates, the equation of interpolation for the cell face properties can be written as follows:

$$\varphi_f = \alpha_d\varphi_D + \alpha_u\varphi_U + \alpha_{uu}\varphi_{UU} \quad (12)$$

Where α_d, α_u, and α_{uu} are the length factors for downstream, upstream, and up-upstream, respectively. For example, these factors will be calculated for the east face with positive flow direction as shown in Fig. 5 as:

$$\alpha_d = \alpha_E = \frac{|\overline{Pe}||\overline{We}|}{(|\overline{Ee}| + |\overline{We}|) * (|\overline{Pe}| + |\overline{Ee}|)} \quad (13)$$

$$\alpha_u = \alpha_P = \frac{|\overline{Ee}||\overline{We}|}{(|\overline{Pe}| - |\overline{We}|) * (|\overline{Pe}| + |\overline{Ee}|)} \quad (14)$$

$$\alpha_{uu} = \alpha_W = \frac{|\overline{Pe}||\overline{Ee}|}{(|\overline{Ee}| + |\overline{We}|) * (|\overline{We}| - |\overline{Pe}|)} \quad (15)$$

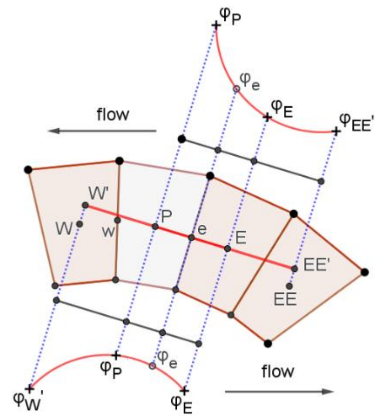


Fig. 5. Quadratic upwind interpolation for convective kinematics (QUICK).

Kholsa and Rubin [46] suggested a solution procedure for the higher-order differencing scheme as QUICK and CDS. This procedure gives more stability needed for the non-orthogonal grides as reported by many other authors such as Dalal, et al. [25] and Ferziger and Peric [4]. This technique is adopted in the current paper as follows:

$$F_e = F_e^L + (F_e^H - F_e^L)^{old} \quad (16)$$

Where the superscript L referred to the lower order differencing scheme (upwind) and the superscript H referred to the higher-order scheme. The second term in the equation is calculated from the previous iteration and stored as a source term in the numerical code.

2.3. Test cases.

The previous discretization schemes will be tested for two test cases as the application of non-orthogonal grids and generalized grids are as follows:

2.3.1. Test case (1)

The polar cavity or curved cavity is a very important test case to verify the general coordinates. The grids in this test case are challenging because of the highly skewed flow as shown in Fig. 6. Fuchs and Tillmark [32] gave experimental and numerical study for this case with two different Reynolds numbers 60, and 350. The Reynold number was calculated based on the tangential velocity for the inner rotating wall and the depth length (the difference between the outer and inner radii). The annulus sector has an angle, θ=1 radian, inner radius (r=475 mm), and outer radius (R=950 mm).

All walls are stationary except the inner curved wall is rotating with constant clockwise angular velocity.

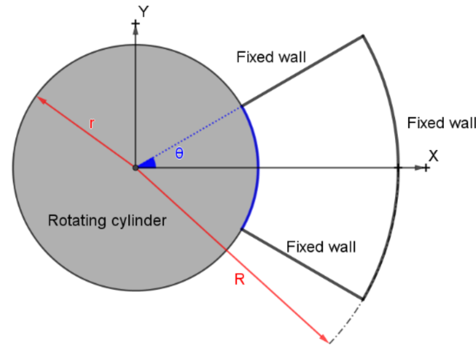


Fig. 6. Curved cavity geometry.

2.3.2.Test case (2)

Lid-driven cavity (skew cavity) flow is a simple case for a non-orthogonal domain with inclination angle β^O on the horizontal. The shape of the skew cavity is represented by a parallelogram as shown in Fig. 7; thus, all grid lines make an angle of β^O.

A comparison is presented with the benchmark solution given by Demirdžić, et al. [18] at $Re = 100$, β=45^O with side length (l=1 m) and upper wall velocity (lid driven velocity) U_{lid}=1. While all other walls are stationary.

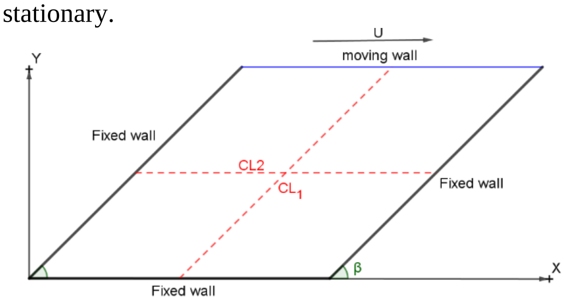


Fig. 7. Skewed cavity geometry.

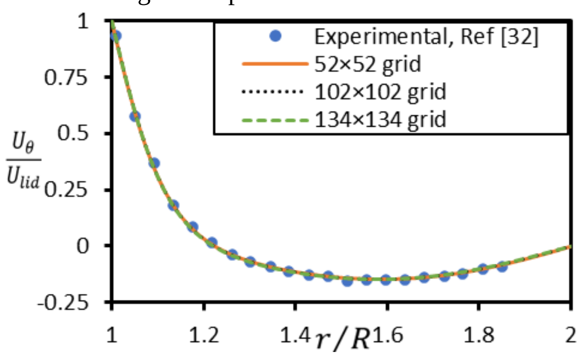
3 Result and Discussion

3.1. Test Case 1

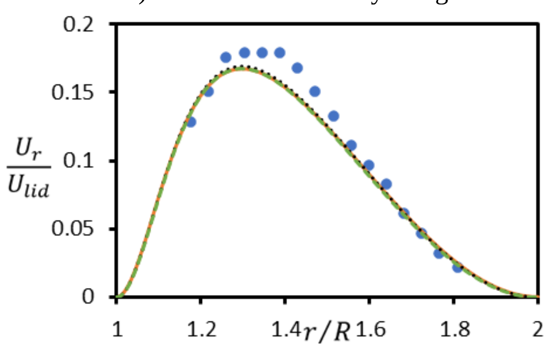
Examining the generalized code based on Cartesian coordinates is applied on the curved cavity with two different Reynolds numbers (60 and 350) and compare the radial and transverse velocity components with experimental data given by Fuchs and Tillmark [32] at (θ=-20, -10, 0, +10, +20 °).

3.1.1. Grid independent study

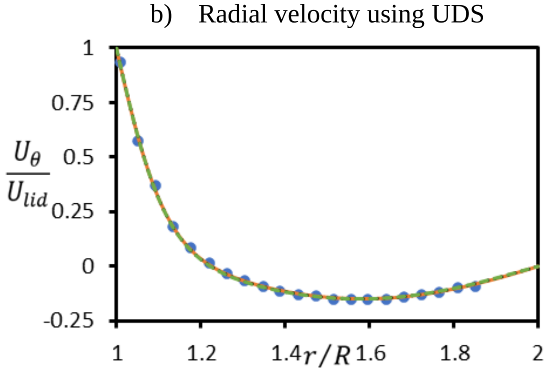
Grid-independent study for the two Reynolds number is shown in Figs. 8 and 9 , For Re=60, as shown in Fig. 8, the grid size of (52×52) gives very close results with the other two fine grids, while the time needed for this grid size is less than the other grids as depicted in Table 1 for all the discretization schemes. The running time required by the CPU for the upwind differencing scheme and the number of iterations is less than the other three schemes. Fuchs and Tillmark [32] presented a numerical solution based on a finite difference scheme and reported that for the small Reynolds number (Re=60) the coarse grids and the upwind difference schemes give a good numerical result. For Re=350, as shown in Fig. 9, the grid-independent study for all the discretization schemes was introduced for both the radial and transverse velocity components. The upwind differencing scheme reaches the grid-independent at (102×102), while the other discretization schemes (upwind with numerical diffusion, QUICK, and CDS) reach the grid-independent at the coarse grid (52×52). The time needed and the iteration numbers to give an accepted solution result for the upwind is greater than the other three schemes as shown in Table (2). Many authors gave the numerical solution at different grid sizes without introducing a grid-independent study as Fuchs and Tillmark [32], Zang and Street [35], Lei, et al. [37] and, Kooshkbaghi and Lessani [30]. While Shu, et al. [47] introduced three grid sizes (49×49, 65×65, and 81×81) with a notice that as the grid number increases the solution gives improved results.



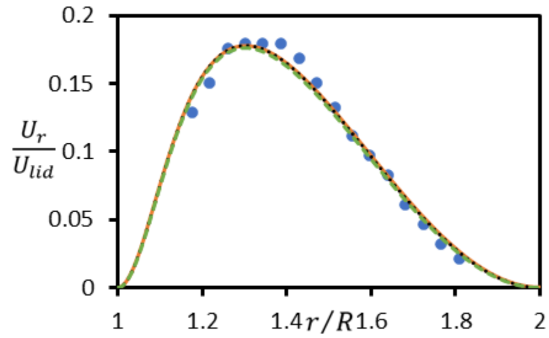
a) Transverse velocity using UDS



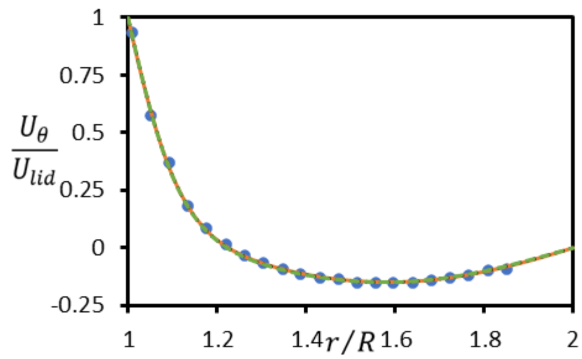
b) Radial velocity using UDS



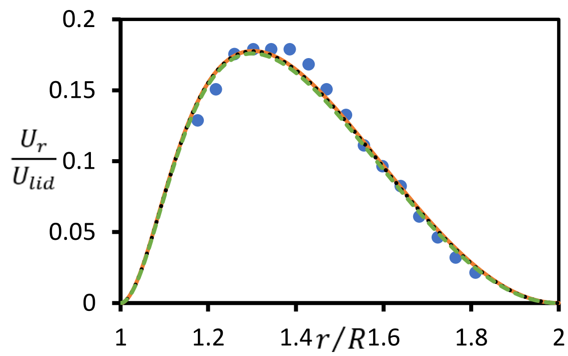
c) Transverse velocity using UDS-ND



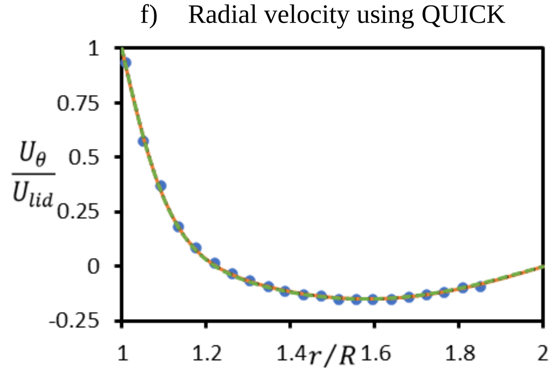
d) Radial velocity using UDS-ND



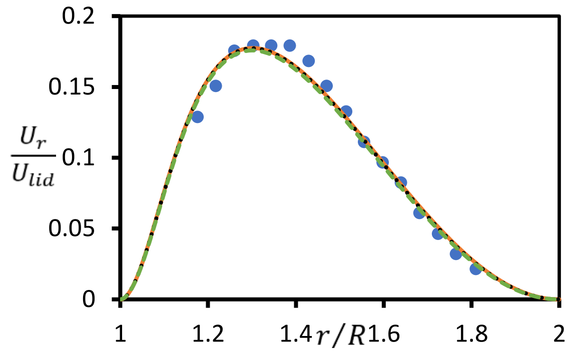
e) Transverse velocity using QUICK



f) Radial velocity using QUICK

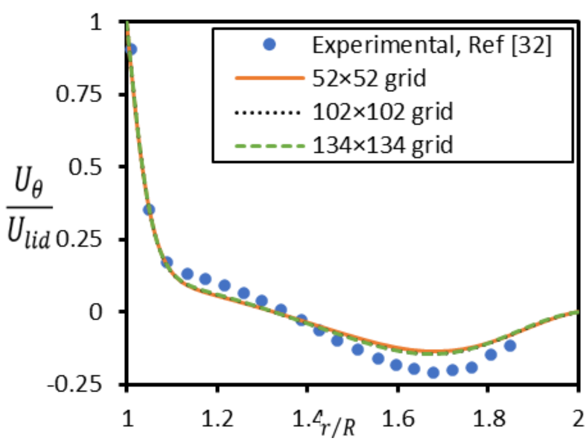


g) Transverse velocity using CDS

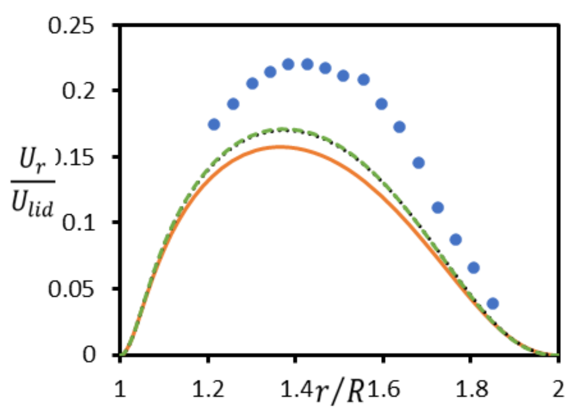


h) Radial velocity using CDS

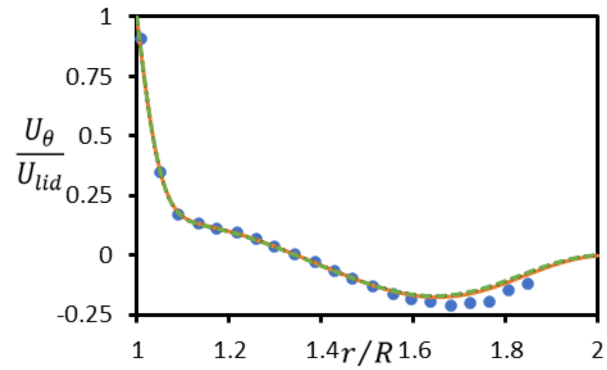
Fig. 8. Grid independent study for curved cavity at Re=60 and θ=+10º.



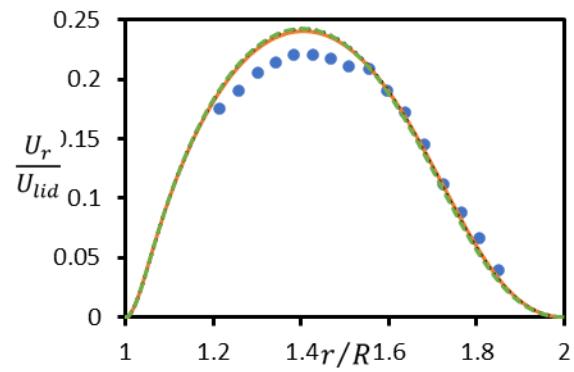
a) Transverse velocity using UDS



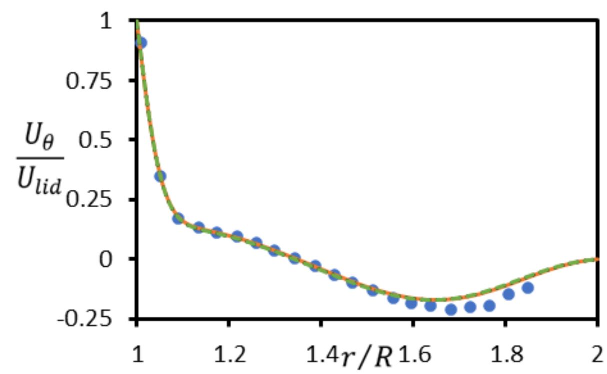
b) Radial velocity using UDS



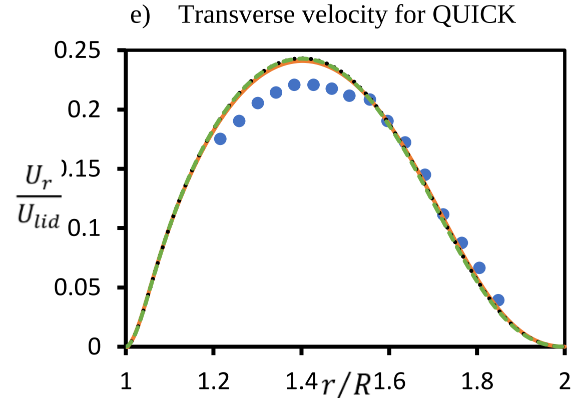
c) Transverse velocity using UDS-ND



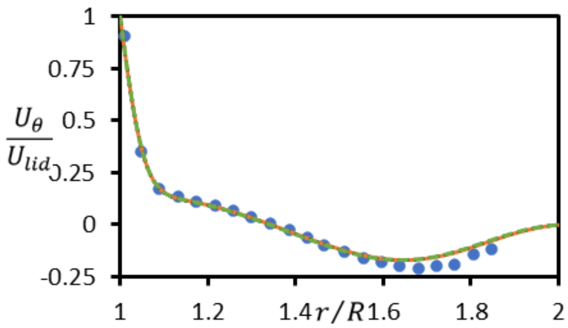
d) Radial velocity using UDS-ND



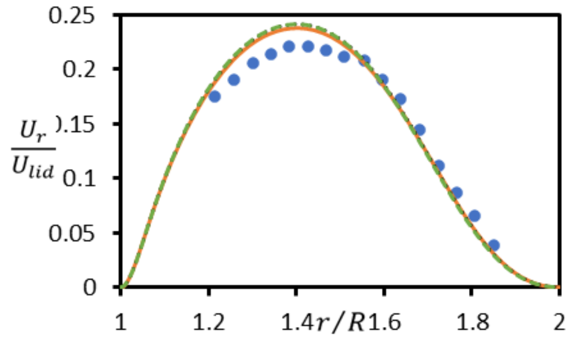
e) Transverse velocity for QUICK



f) Radial velocity for QUICK



g) Transverse velocity using CDS



h) Radial velocity using CDS

Fig. 9. Grid independent study for curved cavity at Re=350 and θ=+10º

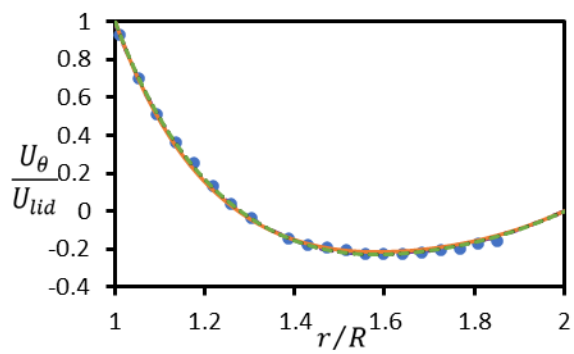
Table 1. Computational time and number of iterations for different discretization schemes at different grid sizes for a curved cavity at Re=60.

Number of grids	UDS		UDS–ND		QUICK		CDS	
	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations
52×52	5.903	713	6.073	753	6.381	753	6.012	752
102×102	53.202	1735	56.192	1777	59.588	1777	54.906	1776
134×134	123.576	2305	128.25	2338	131.074	2338	127.77	2338

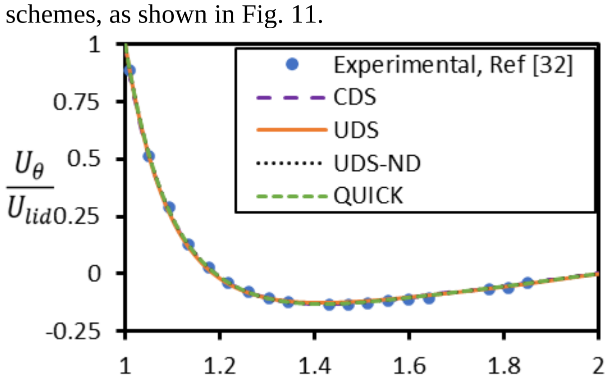
3.1.2. Comparison between different discretization schemes

The difference between the upwind differencing scheme UDS, upwind with numerical diffusion UDS-ND, QUICK, and CDS are illustrated in Figs. (10–13) for the two Reynolds numbers.

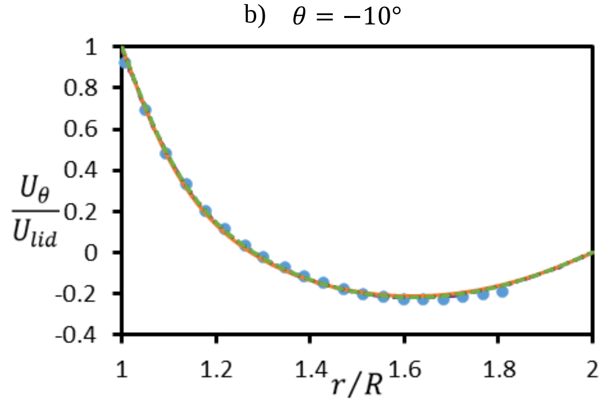
For Re=60, all discretization schemes are very close to each other and with the experimental data of Fuchs and Tillmark [32] for the transverse velocity at all angular positions as shown in Fig. 10. While for the radial velocity, there is a noticed difference between the upwind differencing scheme with the other three schemes, as shown in Fig. 11.



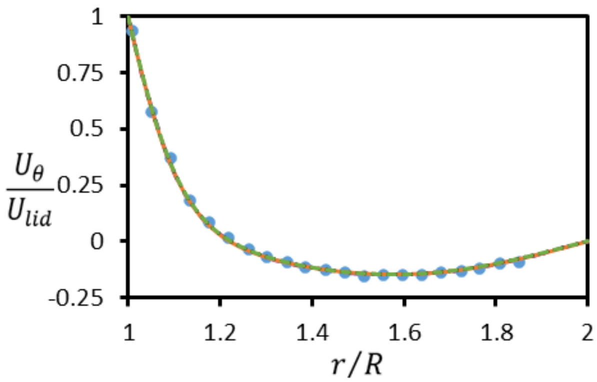
b) $\theta = -10°$



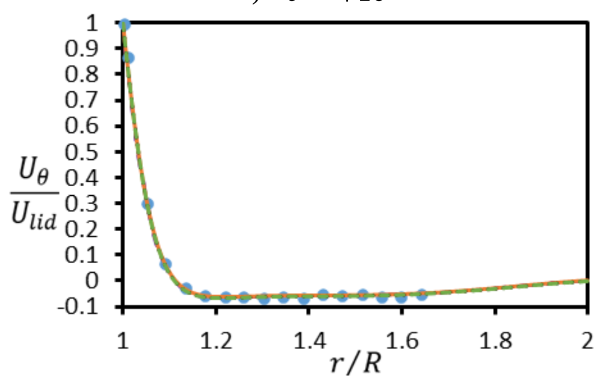
a) $\theta = -20°$



c) $\theta = 0°$

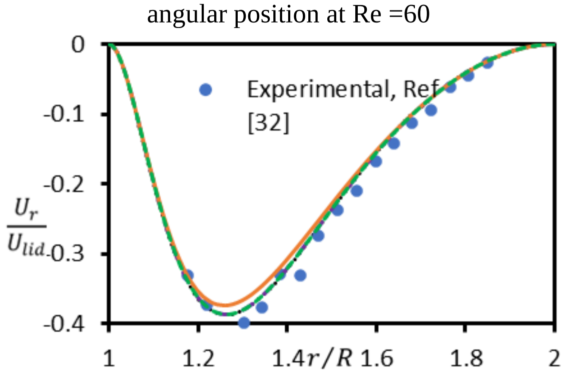


d) $\theta = +10°$

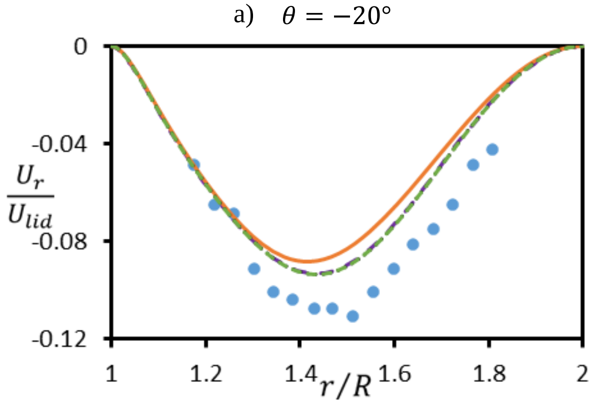


e) $\theta = +20°$

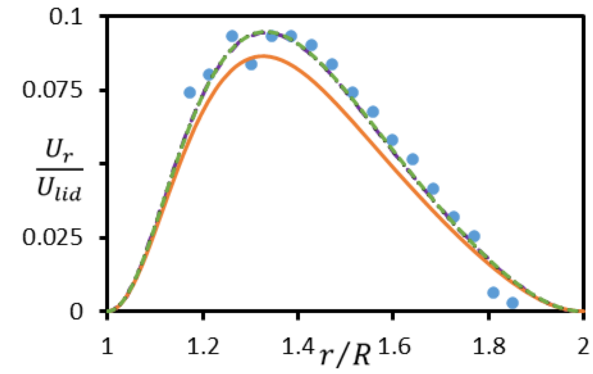
Fig. 10. Effect of different discretization schemes on the transverse velocity component at different angular position at Re =60



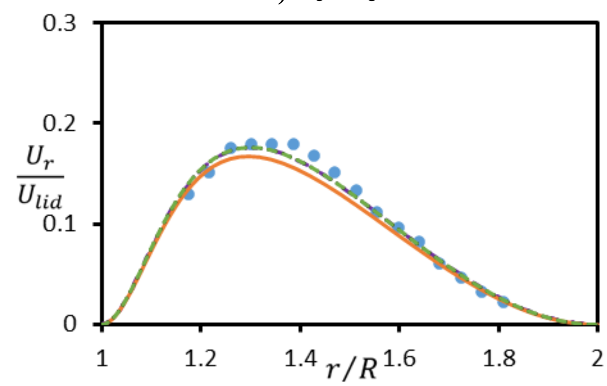
a) $\theta = -20°$



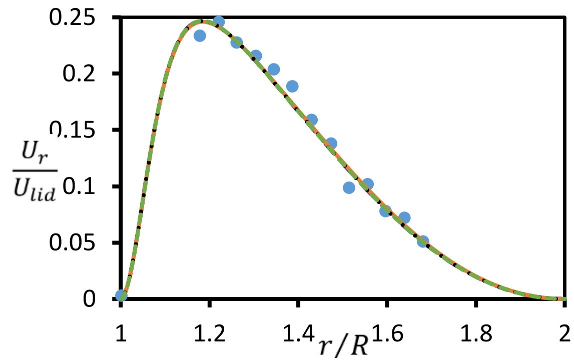
b) $\theta = -10°$



c) $\theta = 0°$



d) $\theta = +10°$

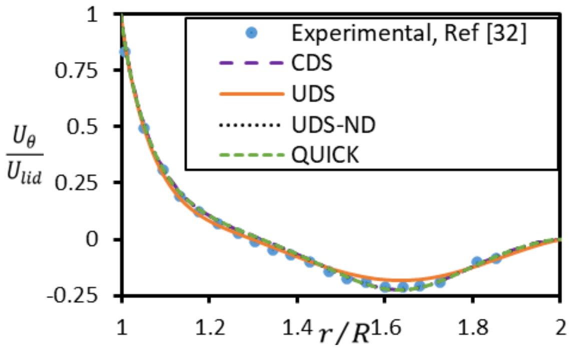


e) $\theta = +20°$

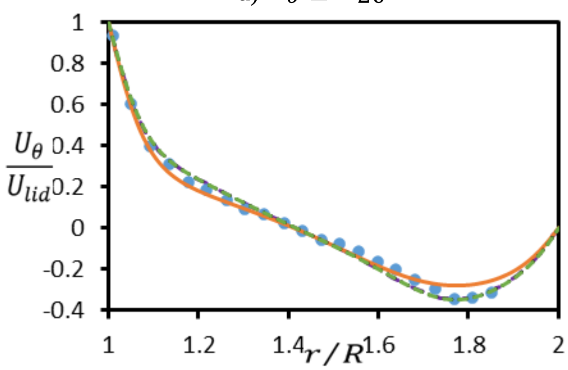
Fig. 11. Effect of different discretization schemes on the radial velocity component at different angular position at Re = 60.

All schemes give accepted predictions for this test case at a low Reynolds number, while the upwind differencing scheme has a lower accuracy at some positions. The skewness of the flow in some positions as mentioned in Raithby [5] may be the better explanation for the bad accuracy of the upwind differencing scheme. The computational time in this case for all schemes, at the grid-independent, is very close, as shown in Tables 1. The better scheme is the upwind with numerical diffusion as it gives better accuracy than the upwind and gives the same accuracy as the QUICK and CDS while the time is less than these two schemes.

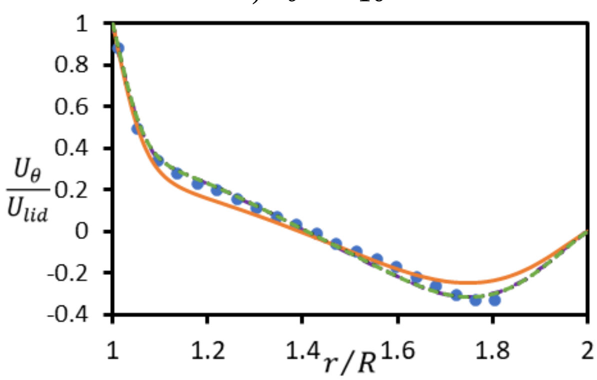
As the Reynolds number increased to 350 the accuracy of all schemes is reduced. The present study is for a two-dimension problem while the effect of the three-dimension domain may be of a great impact on the difference between the present study and the experimental data as stated in many previous studies, as Fuchs and Tillmark [32], Zang and Street [35] and Shu, et al. [47]. The transverse velocity at all five positions shown in Fig. 12, all schemes give the same trend as the experimental data. The upwind differencing scheme at all positions except at (θ=20) has a greater difference than the other schemes, while the radial component shown in Fig. 13 the difference between the upwind differencing scheme and other schemes is high.



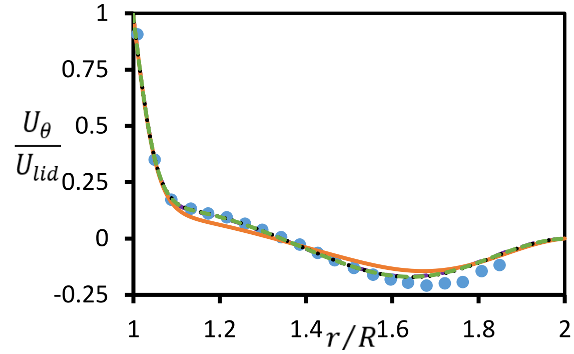
a) $\theta = -20°$



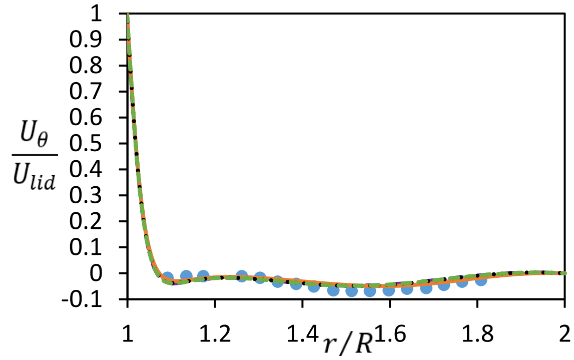
b) $\theta = -10°$



c) $\theta = 0°$

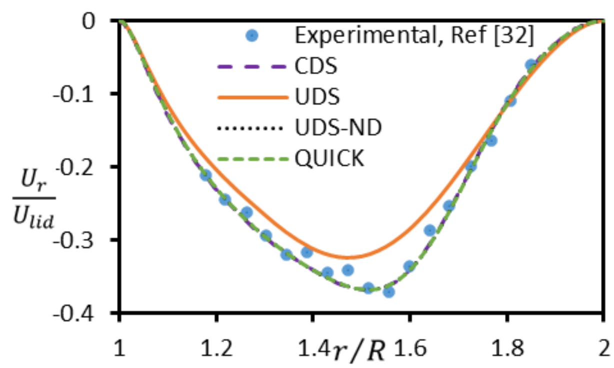


d) $\theta = +10°$

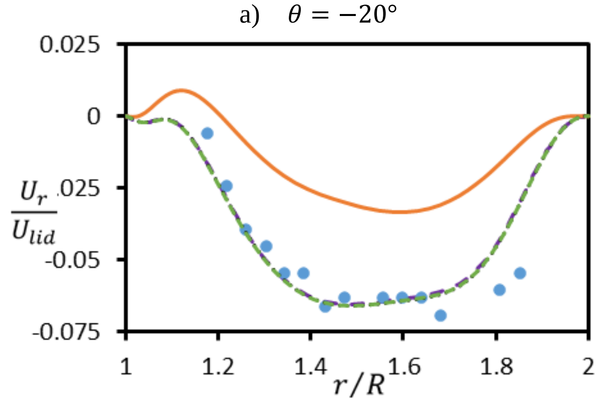


e) $\theta = +20°$

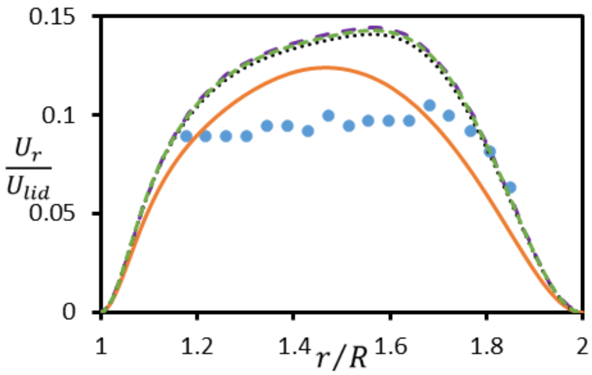
Fig. 12. Effect of different discretization schemes on the transverse velocity component at different angular position at Re =350.



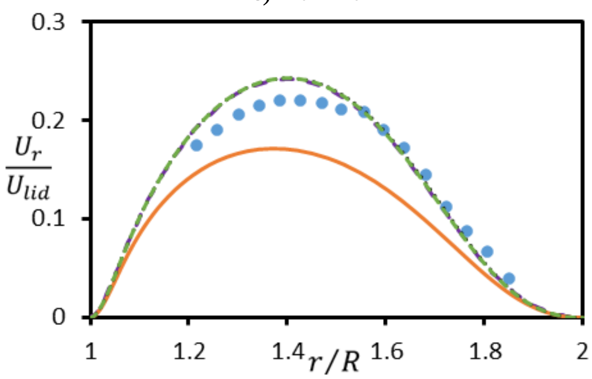
a) $\theta = -20°$



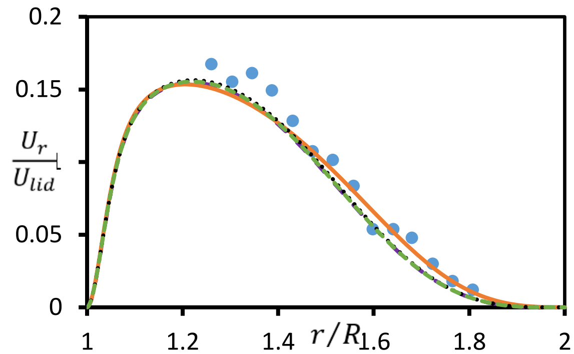
b) $\theta = -10°$



c) $\theta = 0°$



d) $\theta = +10°$



e) $\theta = +20°$

Fig. 13. Effect of different discretization schemes on the radial velocity component at different angular position at Re=350.

Based on the results presented in Table 2, all schemes reach the grid-independent study seven times faster than the upwind differencing scheme. From Figs. (10–13) it's not recommended to use the upwind differencing scheme with the skewed flow.

Table 2. Computational time and number of iterations for different discretization schemes at different grid sizes for a curved cavity at Re=350.

Number of grids	UDS		UDS–ND		QUICK		CDS	
	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations
52×52	5.539	696	6.866	831	6.853	827	6.58	826
102×102	49.02	1626	53.483	1754	54.801	1751	52.73	1751
134×134	113.586	2220	118.79	2303	125.317	2303	125.108	2302

3.1.3. Statistical Analysis

The comparison between different discretization schemes, shown in Figs. (10–13), can show that, the upwind differencing scheme is the worst scheme compared with the others scheme. But the difference between UDS-ND, QUICK, and CDS is very small and can't be observed from the figures. Therefore, some statistical parameters are calculated to evaluate the performance of these schemes in the curved cavity. The Absolute Average Relative Percentage Error (AAPE) can be calculated as,

$$AAPE = 100 \times \frac{1}{N}\sum_{i=1}^{N}|E_i| \qquad (17)$$

Where N is the number of data points and E_i is the relative error at any position (i), which calculated from the relation:

$$E_i = \left[\frac{(U)_{act,i} - (U)_{est,i}}{(U)_{act,i}}\right] \qquad (18)$$

Where $(U)_{act}$ is the actual velocity from the experimental data, $(U)_{est}$ is the estimated velocity from the numerical.

The Root Mean Square Error (RMSE) can be calculated from:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[(U)_{act,i} - (U)_{est,i}\right]^2} \qquad (19)$$

The Correlation Coefficient (R) is

$$R = \frac{\sum_{i=1}^{N}\left[(U)_{act,i} - \overline{(U)}_{act}\right]\left[(U)_{est,i} - \overline{(U)}_{est}\right]}{\sqrt{\sum_{i=1}^{N}\left[(U)_{act,i} - \overline{(U)}_{act}\right]^2}\sqrt{\sum_{i=1}^{N}\left[(U)_{est,i} - \overline{(U)}_{est}\right]^2}} \qquad (20)$$

Where $\overline{(U)}_{act}$ is the average of actual velocity, and $\overline{(U)}_{est}$ is the average of estimated velocity.

Standard Deviation (SD)

$$SD = \left[\frac{N\sum_{i=1}^{N} E_i^2 - \left(\sum_{i=1}^{N} E_i\right)^2}{N^2}\right]^{\frac{1}{2}} \qquad (21)$$

All the previous statistical parameters are tabulated in Tables 3 and 4 for transverse and radial velocity components, respectively, at Re=350.

The error of upwind with numerical diffusion, Quick and CDS is very close to each other and less than the upwind differencing schemes error represented by AAPE and RMS as shown in Tables 3 and 4. QUICK is better than CDS and upwind differencing scheme in this test case, this conclusion agreed with old numerical studies for non-orthogonal grids as in Huang, et al. [6], Choi, et al. [24], Roychowdhury, et al. [21] and, Paramane and Sharma [26].

Upwind with numerical diffusion is the better scheme than the other schemes (has smallest AAPE, RMS and has the largest R) also gives a grid-independent at a coarse grid with less time as shown in Fig. 11 and Table 2. Also, it's a lower order convective scheme as the original upwind differencing scheme which gives a more stable solution than the higher-order ones as QUICK and CDS.

Table 3. Statistical parameters for transverse velocity in a curved cavity at Re=350

	APPE	RMS	R	SD
UDS	25.46664579	0.033716443	0.993949726	0.385060598
UDS-ND	22.74733212	0.022909984	0.99641833	0.402398743
QUICK	23.37716593	0.023205613	0.996313325	0.414064691
CDS	23.8307308	0.023513988	0.996196461	0.421095428

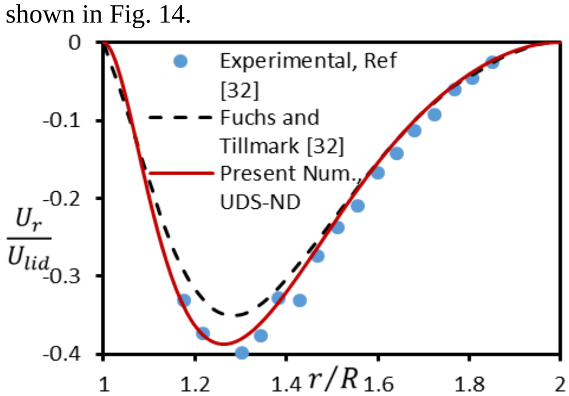
Table 4. Statistical parameters for radial velocity in a curved cavity at Re=350

	APPE	RMS	R	SD
UDS	22.0034606	0.027660256	0.988904576	0.300988579
UDS-ND	13.33872047	0.016226453	0.995347265	0.251521921
QUICK	13.78733192	0.016775706	0.994972828	0.258583618
CDS	13.98443056	0.017133188	0.994653609	0.258176129

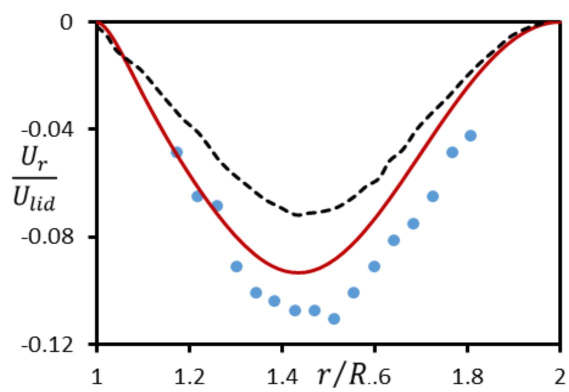
3.1.4. Comparison with other numerical studies:

A comparison with published numerical solution based on different numerical techniques or different governing equations will be presented in this section to show the difference between the present work and other studies.

Although the numerical solution for the lower Reynolds number gives a very good prediction for the two components of velocity, the numerical results given by Fuchs and Tillmark [32] at the first two positions (θ=-20, -10) shows a large difference with their experimental data. The present work gives a better prediction for the velocity component especially for the radial component at θ=-10 than their work, as shown in Fig. 14.



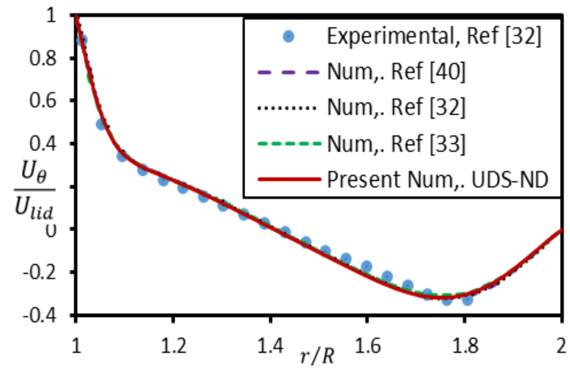
a) $\theta = -20$



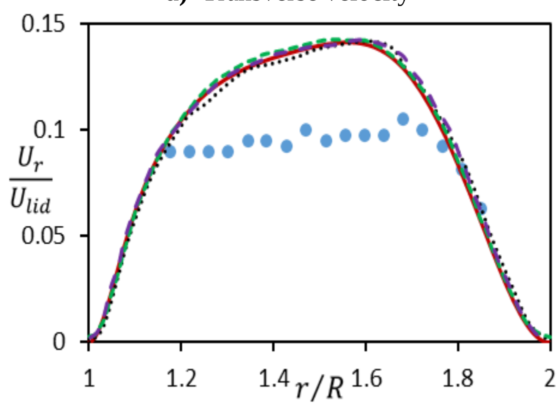
b) $\theta = -10$

Fig. 14. Comparison predicted radial velocity and numerical solution based in polar coordinates with (ψ-ω) formulation given by [32] at Re=60 at different positions

Figure 15 shows that the results of the present study are very close with the numerical solution of many works which used the polar coordinates with stream function - vorticity formulations at Re=350. The results show a small difference between the present work and that obtained by stream function-vorticity formulations. However, the stream function - vorticity formulations and polar coordinates are not suitable for general-purpose codes.



a) Transverse velocity



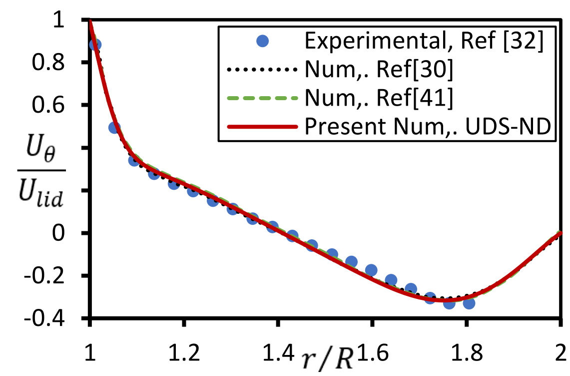
b) Radial velocity

Fig. 15. Comparison between published numerical solution based in polar coordinates with (ψ-ω) formulation and the present study at Re = 350 and θ=0º.

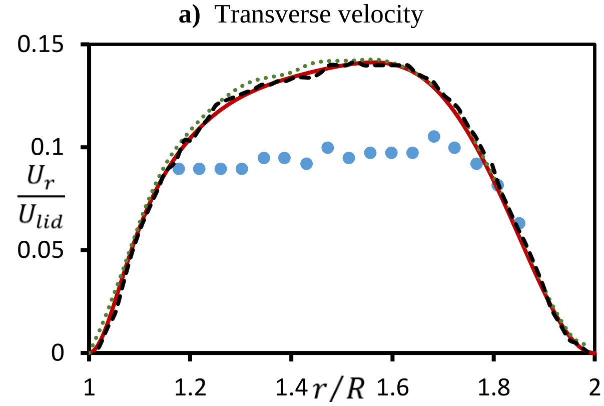
The solution based on body-fitted coordinates transformation for the Cartesian governing equations given by Kooshkbaghi and Lessani [30] and Sen and Kalita [41] are compared with the present study, as presented in Fig. 16. There is a good agreement between the present study and those given by Kooshkbaghi and Lessani [30] and Sen and Kalita [41]. The body-fitted coordinates are more general than the polar coordinates with stream function-vorticity formulations, but the transformed equation has more terms than the original Cartesian equations. These terms will increase the computation times needed for the solution and increase the programming effort. Also, the non-orthogonal grids give a chance for the unphysical solution as mentioned by Ferziger and Peric [4].

Darbandi and Vakilipour [39] presented a solution with Cartesian coordinate equations and applied a pressure-weighted upwinding scheme for the unstructured grid. A comparison between the present results and that of Darbandi and Vakilipour [39] is depicted in Fig. 17. It can be seen from this figure that the present results are in close agreement with those of Darbandi and Vakilipour [39].

From the previous discussion, using the finite volume method for a Cartesian coordinate will give more generality than the polar coordinates and body-fitted coordinates. The upwind differencing scheme with numerical diffusion is the better discretization scheme for the convective term as it gives better accuracy than other schemes and gives a minimum computation time for the skewed velocity flows.

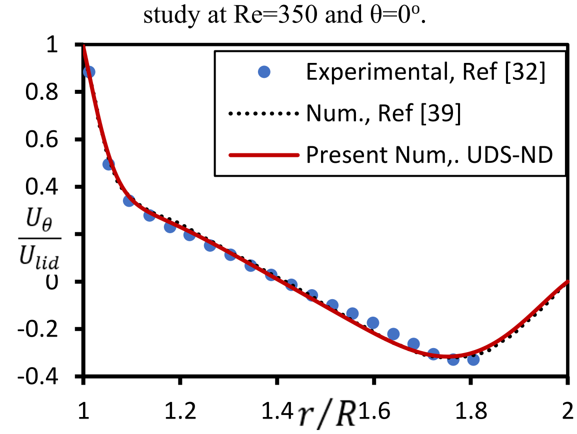


a) Transverse velocity

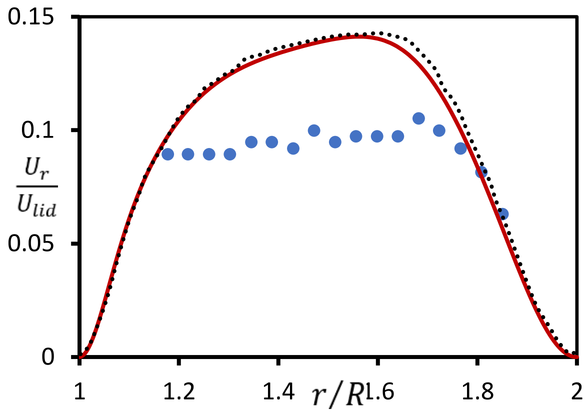


b) Radial velocity

Fig. 16. Comparison between published numerical solution based in Cartesian coordinates with body fitted coordinates transformation and the present study at Re=350 and θ=0º.



a) Transverse velocity



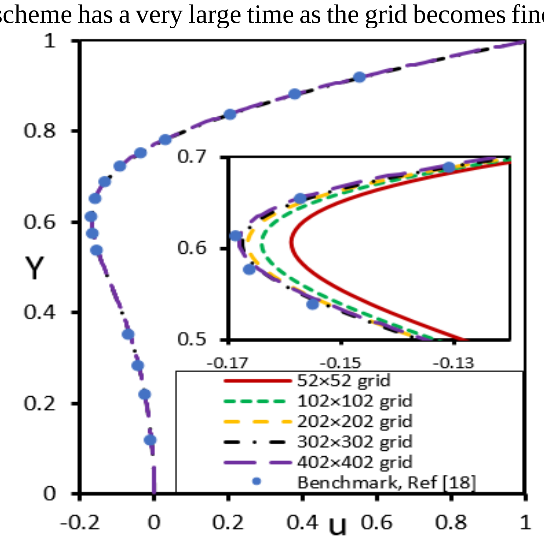
b) Radial velocity

Fig. 17. Comparison between published numerical solution based in Cartesian coordinates with finite element method and the present study at Re=350 and θ=0º.

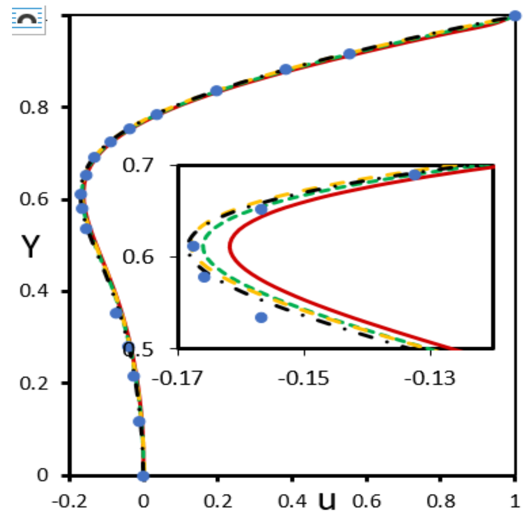
3.2 Test case 2

3.2.1 Grid independent study

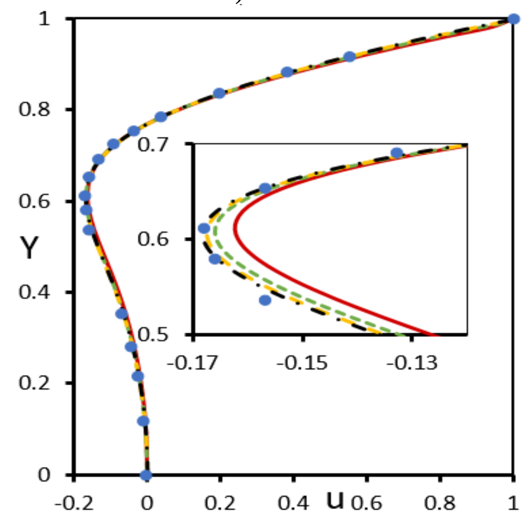
A grid-independent study for the four schemes had been done to the skewed cavity to test the effect of the discretization scheme on the non-orthogonal grids. Figure 18 shows that the upwind differencing scheme requires a fine grid (302×302) to give a grid-independent solution, as shown in Fig. (18-a). All other schemes reach the grid-independent at a coarse grid than the upwind (202×202) as shown in Figs. 18(b to d). The CPU time necessary for the different schemes at different grid sizes is shown in Table 5. The number of iteration and the CPU time is very close between all discretization schemes. The CDS and upwind with numerical diffusion are the better schemes from both grid-independent and CPU time. Paramane and Sharma [26] reported that the QUICK scheme has a very large time as the grid becomes finer.



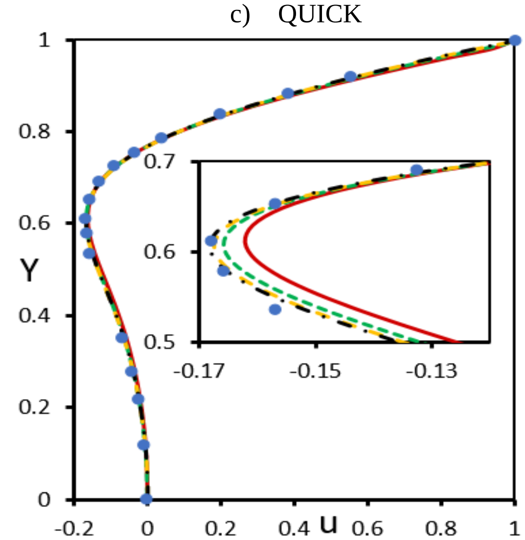
a) **UDS**



b) UDS-ND



c) QUICK



d) CDS

Fig. 18. Effect of different grid size on the horizontal velocity component for different discretization schemes.

Table 5. computational time and number of iterations for different discretization schemes at different grid sizes for skewed cavity

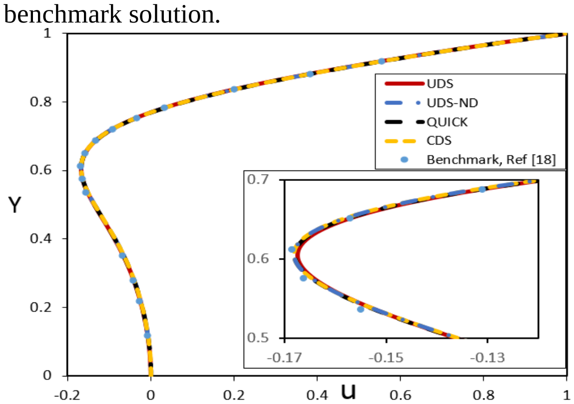
Number of grids	UDS		UDS–ND		QUICK		CDS	
	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations	TIME (sec.)	No. of iterations
52×52	5.533	704	5.782	719	6.318	725	5.661	718
102×102	71.876	2779	82.336	2804	89.486	2803	83.299	2803
202×202	395.685	4641	399.379	4662	452.818	5000	395.232	4661
302×302	1699.379	8661	1682.843	8686	2126.94	10000	1671.463	8686

Table 6. Maximum and minimum stream function, ψ and their locations

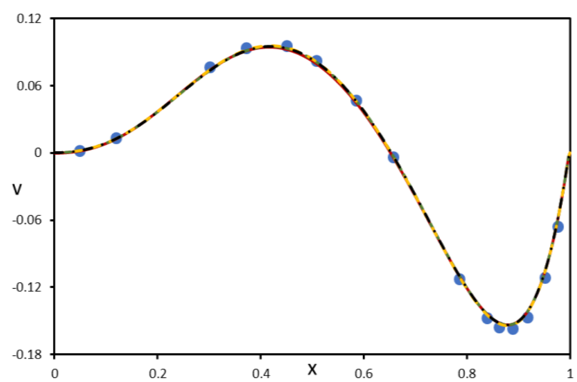
Study	Maximum streamline			Minimum streamline		
	Value×10^5	Location		Value×10^2	Location	
Demirdžiü et al. [18]	3.68310	0.33867	0.14308	-7.02260	1.11	0.54638
Present study (UDS)	3.51790	0.33818	0.14318	-6.97718	1.11525	0.55025
present study (UDS-ND)	3.68336	0.33818	0.14318	-6.99917	1.11025	0.54625
present study (QUICK)	3.51076	0.33818	0.14318	-7.01040	1.11288	0.54788
present study (CDS)	3.38390	0.33818	0.14318	-6.99918	1.11525	0.55025

3.2.1. Comparison with benchmark solution

The present results for vertical and horizontal velocity components are compared with the benchmark solution given by Demirdžiü et al. [18], as shown in Fig. 19. It can be seen from this figure that all schemes, when a grid-independent solution is reached, give nearly the same results. The magnitude of maximum and minimum values of stream function along with their locations are given in Table 6 and the stream function contours are presented in Fig. 20. It can be seen from these results that all the tested schemes produce acceptable results as compared with the benchmark solution.



a) Horizontal velocity



b) Vertical velocity

Fig. 19. Effect of different discretization schemes on the velocity component in skewed cavity.

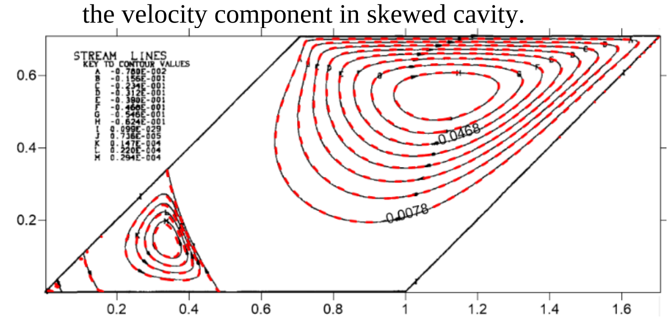


Fig. 20. Comparison between predicted streamlines (dashed lines) and benchmark solution given by Ref. [18] (solid lines).

3.2.2. Statistical Analysis

After reaching the grid independence, all schemes give nearly the same results. Statistical analysis results are presented in Tables 7 and 8 for both horizontal and vertical velocity components. For the horizontal velocity component, the UDS-ND gives the best results. On the other hand, UDS and QUICK schemes give the best results for the vertical velocity component while the upwind with numerical diffusion and CDS are close to each other in accuracy. However, it was previously shown in Table 5 that the upwind differencing requires approximately twice the number of iteration and 425% of the computational time of the upwind with numerical diffusion. Thus, the upwind differencing scheme is not recommended for flow problems with non-orthogonal grids. Paramane and Sharma [26] showed that the accuracy of the CDS and QUICK is less than the upwind differencing scheme. But their study had been done for a smaller number of grids and the accuracy was calculated only from the mean absolute relative error for the stream functions. Overall, the UDS-ND is recommended for non-orthogonal grids.

Table 7. Statistical analysis for horizontal velocity in skewed cavity for all discretization schemes

	APP	APPE	RMS	R	SD
UDS	1.358749227	2.62814030	0.001760236	0.9999658020	0.031767379
UDS-ND	1.307173018	2.29146904	0.002114573	0.999987910	0.030129066
QUICK	1.307150356	2.34460594	0.001957263	0.999976238	0.029105899
CDS	1.310842812	3.00676636	0.002121171	0.999957583	0.037139075

Table 8. Statistical analysis for vertical velocity in a skewed cavity for all discretization schemes.

	APP	APPE	RMS	R	SD
UDS	4.061798749	0.52979984	0.00308591	0.9999523727	0.021586863
UDS-ND	5.207613295	0.67925391	0.003040123	0.999937540	0.056306557
QUICK	4.307150008	0.56180217	0.002630667	0.999936989	0.040670632
CDS	5.236000757	0.68295662	0.00304057	0.999937184	0.057161916

4.Conclusions

Four different discretization schemes were evaluated in terms of accuracy, the number of iterations, and computation time for two test cases. These schemes are upwind differencing scheme (UDS), upwind differencing scheme with numerical diffusion (UDS-ND), central differencing scheme (CDS), and Quadratic upwind interpolation for convective kinematics (QUICK). The comparison between the selected schemes for highly skewed flow in curved cavity indicated that the upwind differencing scheme with numerical diffusion is the best choice in terms of accuracy and computational cost. In addition, the present study indicates the current procedure which is more suitable for general purpose codes can produce computational results which are in close agreement with those obtained from body fitted and polar coordinates system. For the case that included non-orthogonal grids in the skewed cavity, all the tested schemes produce close results when they were compared with the benchmark solution. However, the upwind differencing scheme with numerical diffusion requires fewer number of iteration and shorter computational time.

Nomenclature

φ	The generalized dependent variable
μ	The kinematic viscosity [$m^2 s^{-1}$]
q_φ	The source term
$\boldsymbol{n}$	The unit normal vector outward from the center of the cell face
f=e, w, n, s	The cell face center's properties

Subscripts

L	The lower order differencing scheme (upwind)
H	the higher-order scheme

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