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Simplified method to calculate the coefficients in one form of linear difference equations knowing the coefficients in the other form.

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Abstract:

The dynamic behavior of linear systems viewed in discrete-time is usually described by linear difference equations. An n-th order linear difference equation can be constructed using the difference of the output of the system up to the n-th order difference, or using the values of the output of the system at n discrete instants of time.

According to the problem to be solved one form of the difference equation may be more convenient than the other. The conversion from one form of the difference equation to the other is a problem of calculating the coefficients in the needed form knwoing the coefficients in the other form.

In this paper we give a simplified method to calculate the coefficients in one form of the n-th order difference equation knowing the coefficients in the other form. To illustrate the proposed method an example of 7-th order equation is worked.

I- Problem Formulation

The dynamic behavior of a system viewed in discrete-time is usually described by difference equations. An n-th order linear difference equation of the form:

$$\Delta^n Y(K) + a_1(k)\,\Delta^{n-1} Y(k) + \ldots + a_n(k)\,Y(k) = u(k) \qquad (1)$$

$\Delta^r Y(k)$ = The r-th order difference of Y(k) computed by the forula; [1]

$$\Delta^r Y(k) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} Y(k + r - i) \qquad (2)$$

In many cases it is more convenient to write the difference equation (1) in the form; [2]

$$Y(k+n) + b_1(k)\,Y(k+n-1) + \ldots + b_n(k)\,Y(k) = u(k) \qquad (3)$$

Given one form of the differnce equation it is required to obtain the other form. This can be accomplished by computing the coefficients $b_i(k)$ knowing $a_i(k)$, or by computing $a_i(k)$ knowing $b_i(k)$; [1].

II- From $a_i(k)$ to $b_i(k)$.

For the n-th order difference equation the relationship between the coefficients $b_i(k)$ and $a_i(k)$ can be described by the following matrix equation:

$$\begin{bmatrix} b_1(k) \\ \vdots \\ b_i(k) \\ \vdots \\ b_n(k) \end{bmatrix} = \begin{bmatrix} c_{11}(k) & \cdots & c_{1n}(k) \\ \vdots & & \vdots \\ \cdots & C_{ij}(k) & \cdots \\ \vdots & & \vdots \\ (k) & \cdots & c_{nn}(k) \end{bmatrix} \begin{bmatrix} a_1(k) \\ \vdots \\ a_i(k) \\ \vdots \\ a_n(k) \end{bmatrix} + \begin{bmatrix} d_1(k) \\ \vdots \\ d_i(k) \\ \vdots \\ d_n(k) \end{bmatrix} \qquad (4)$$

The coefficients $c_{ij}(k)$ and $d_i(k)$ can be computed by the following formulas:

$$c_{ij}(k) = \begin{cases} 0 & i < j, \quad i > n \\ 1 & i = j \\ (-1)^{i+j} \left(|c_{i,j+1}| + |c_{i+1,j}| \right) & i > j \end{cases} \qquad (5)$$

$$d_i(k) = (-1)^i \left(|c_{i1}| + |c_{i+1,1}| \right) \qquad (6)$$

In simplified notation the mtarix equation (4) can be written in the form:

$$\underline{B}(k) = \underline{C}(k)\,\underline{A}(k) + \underline{D}(k) \qquad (7)$$

$\underline{B}(k)$ = The required nxl column matrix;

$\underline{A}(k)$ = The known nxl column matrix;

$\underline{C}(k)$ = An nxn square matrix to be constructed;

$\underline{D}(k)$ = An nxl column matrix to be constructed.

III- Construction procedures for the matrics $\underline{C}(k)$ and $\underline{D}(k)$.

The matrix $\underline{C}(k)$ is a lower triangular matrix and can be constructed as follows:

For a seventh-order equation proceed as follows:

1. Construct a 7 x 7 matrix with all the diagonal elements equal to unity;
2. Put all elements above the diagonal equal to zero;
3. In the last row put ones with changing sign from right to left;

4. Compute the remaining elements of the matrix going from right to left and from below upwards by summing the absolute values of the two elements at right in the same row and the row in below changing the sign to the opposite at each step;
5. Go upwards to complete the matrix.

For our example of 7-th order equation, we obtain:

$$\underline{C}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 15 & -5 & 1 & 0 & 0 & 0 & 0 \\ -20 & 10 & -4 & 1 & 0 & 0 & 0 \\ 15 & -10 & 6 & -3 & 1 & 0 & 0 \\ -6 & 5 & -4 & 3 & -2 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

After computing the matrix $\underline{C}(k)$ we can compute the column matrix $\underline{D}(k)$ as follows:

1. The value of the first element from above in the column matrix $\underline{D}(k)$ is the sum of the absolute values of the two elements from above in the first column $\underline{C}(k)$;

2. The sign of the first element in $\underline{D}(k)$ from above is always negative;

The construction of the matrix $\underline{H}(k)$ can be accomplished by the following procedure explained on an example of the seventh order equation:

1. Construct the 7 x 7 matrix with all the diagonal elements equal to unity;
2. Put all the elements above the diagonal equal to zero;
3. In the last row, the seventh in this example, put ones;
4. Compute the remaining elements of the matrix going from right to left and from below upwards by summing the values of the two elements at the right in the same row and the one in below;
5. Go upwards to complete the matrix.

For our example of 7-th order equation, we obtain:

$$\underline{H}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 15 & 5 & 1 & 0 & 0 & 0 & 0 \\ 20 & 10 & 4 & 1 & 0 & 0 & 0 \\ 15 & 10 & 6 & 3 & 1 & 0 & 0 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The column matrix $\underline{F}(k)$ has all its elements positive.

After computing the matrix $\underline{H}(k$ we can compute the column matrix $\underline{F}(k)$ as follows:

1. The value of the first element from above in $\underline{F}(k)$ is the sum of the two elements from above in the first column of

2. The value of the second element from above in $\underline{F}(k)$ is the sum of the second and the third elements from above in the first column of $\underline{H}(k)$, etc.

3. Go downwards to complete the column matrix.

For our example of 7-th order equation, we obtain:

$$\underline{F}(k) = \begin{bmatrix} 7 \\ 21 \\ 35 \\ 35 \\ 21 \\ 7 \\ 1 \end{bmatrix}$$

From equation (12) we can obtain an expression for $\underline{B}(k)$ knowing $\underline{A}(k)$;

$$\underline{B}(k) = \underline{H}^{-1}(k) \; [\underline{A}(k) - \underline{F}(k)] \tag{13}$$

3. The value of the second element from above in $\underline{D}(k)$ is the sum of the absolute values of the second and the third elements from above in the first column of $\underline{C}(k)$;

4. The sign changes to the opposite in each step.

For our example of 7-th order equation, we obtain:

$$\underline{D}(k) = \begin{bmatrix} -7 \\ 21 \\ -35 \\ 35 \\ -21 \\ 7 \\ -1 \end{bmatrix}$$

IV- From $b_i(k)$ to $a_i(k)$

TO compute $\underline{A}(k)$ knowing $\underline{B}(k)$, we obtain from equation (7):

$$\underline{A}(k) = \underline{C}^{-1}(k) \; [\; \underline{B}(k) - \underline{D}(k) \;] \qquad (8)$$

To avoid the computation of the inverse matrix $\underline{C}^{-1}(k)$ we can use the folowing procedure:

Let the needed transformation from $b_i(k)$ to $a_i(k)$ be described by the matrix equation:

$$\begin{bmatrix} a_1(k) \\ \vdots \\ a_i(k) \\ \vdots \\ a_n(k) \end{bmatrix} = \begin{bmatrix} h_{11}(k) & \cdots & h_{1n}(k) \\ \vdots & \vdots & \vdots \\ \cdots & h_{ij}(k) & \cdots \\ \vdots & \vdots & \vdots \\ h_{n1}(k) & \cdots & h_{nn}(k) \end{bmatrix} \begin{bmatrix} b_1(k) \\ \vdots \\ b_i(k) \\ \vdots \\ b_n(k) \end{bmatrix} + \begin{bmatrix} f_1(k) \\ \vdots \\ f_i(k) \\ \vdots \\ f_n(k) \end{bmatrix} \quad (9)$$

The coefficients $h_{ij}(k)$ and $f_i(k)$ can be computed by the following formulas:

$$h_{ij}(k) = \begin{cases} 0 & i < j, \quad i > n \\ 1 & i = j \\ h_{i,j+1} + h_{i+1,\ j} & i > j \end{cases} \quad (10)$$

$$f_i(k) = h_{i1}(k) + h_{i+1,\ 1}(k) \quad (11)$$

In simplified notation the matrix equation (9) can be written in the form:

$$\underline{\underline{A}}(k) = \underline{\underline{H}}(k)\ \underline{\underline{B}}(k) + \underline{\underline{F}}(k) \quad (12)$$

$\underline{\underline{A}}(k)$ = The required nxl column matrix;

$\underline{\underline{B}}(k)$ = The known nxl column matrix;

$\underline{\underline{H}}(k)$ = An nxn sequare matrix to be constructed;

$\underline{\underline{F}}(k)$ = An nxl column matrix to be constructed.

V- Construction procedures for the matrices $\underline{\underline{H}}(k)$ and $\underline{\underline{F}}(k)$.

The matrix $\underline{\underline{H}}(k)$ is a lower triangular matrix with its diagonal elements equal to unity and with the last row (i=n) consisting of +1's. All the elements of the matrix $\underline{\underline{H}}(k)$ are positive.

VI- Conclusion.

Simple procedures are obtained to relate the coefficients in the two forms of the difference equation. Without doubt these transformation procedures from $a_i(k)$ to $b_i(k)$ and from $b_i(k)$ to $a_i(k)$ as the specified problem requires are very simple and much more easier and quicker to caryy out if compared with usual formulas specially when we deal with high order equations.

References:

[1] **Tsypkin Ya. Z.**
Theory of linear sampled-data systems.
Moscow, 1963, (In Russian).

[2] **Leunberger D. G.**
Introduction to dynamic systems.
1979.
