On Multiple Objective Linear Programming Problems with Fuzzy Rough Coefficients

مشاكل البرمجة الخطية متعددة الأهداف ذات المعاملات الفازية الخشنة

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الخلاصة :

البحث يتناول بالدراسة مشاكل البرمجة الخطية متعددة الأهداف ذات المعاملات الفازية الخشنة. تمت تلك الدراسة بدمج الأعداد الفازية الخشنة بدوال الهدف و كذالك دوال القيود التي تشكل الهيكل العام للمشكلة. و توصلنا إلى تعريف لحل المشكلة تحت الاعتبار من خلال إدخال مفاهيم حلول الكفاءة الفازية. و تم الحصول على الشرط الضروري و الكافي للحل. و أعطينا مثالا عدديا لتوضيح فكرة البحث.

Abstract:

This paper deals with the multiple objective linear programming problems with fuzzy rough coefficients. We consider the problem by incorporating fuzzy rough coefficients into a multiple objective linear programming framework. A solution concept that is attractive from the stand points of feasibility and rough efficiency is specified. A necessary and sufficient condition for such a solution is established. A numerical example is also included for the sake of illustration.

1. Introduction

Making decisions involving multiple objectives is a daily task for a lot of researchers in the more diverse fields. Hence multiple objective decision making problems have defined a very well studied topic in the general area of decision making theory. In particular, multiobjective decision making problems which can be modeled as mathematical programming problems are also well known.

The involvement of different kinds of fuzziness in these problems is an interesting matter. It has received a great deal of work since the early 1980s. This is due to the fact that decision makers have some lack of precision in stating some of the parameters involved in the model.

The multiple objective linear programming problem arises when two or more non comparable linear criterion functions are to be simultaneously optimized over a polyhedral set.

Many researchers are concerned with this subject, (see, for instance, Kuhn and Tucker [6], Tamura and Miura [15], Vangeldere [17] and Zadeh [18]).

Tanaka and Asai [16] formulated multiobjective linear programming problems with fuzzy parameters. This has be done by following the fuzzy decision or minimum operator proposed by Bellman and Zadeh [2] together with triangular membership functions for fuzzy parameters. They considered two types of fuzzy multiobjective linear programming problems. One is to determine the non fuzzy solution and the other is to determine the fuzzy solution. There are different methods for comparison of fuzzy numbers [1,3,4,9,13,14]. One of the most convenient methods is the comparison by integration [1,4,9]. By using the concept of comparison of fuzzy numbers, Maleki and Tata [8]. introduced a new method for solving the fuzzy number linear programming problems.

The notion of rough sets has been introduced by Pawlak [11], and subsequently the algebraic approach to rough sets was studied by Iwinski [5]. The concept of fuzzy sets was investigated by

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Zadeh [19]. A comparison between fuzzy sets and rough sets was introduced by Pawlak [12], where it is shown that these concepts are different. Nanda and Majumdar [10], were introduced the concept of fuzzy rough set.

The main objective of the current paper is to investigate the problem of incorporating fuzzy rough data into a multiple objective linear programming framework. Inspired by the notions of Luhandjula [7], we define a satisfying solution for this problem via two concepts of α-fuzzy rough feasibility and β-fuzzy rough efficiency. A necessary and sufficient condition for a potential action to be a satisfying one is established and some techniques to single out such a solution are described. A numerical example is given for the sake of illustration.

The paper is organized as follows: In section 2, some preliminaries and definitions are given. Section 3 is devoted to the statement of the fuzzy rough multiple objective linear programming problem (P_1) and the concepts of α-fuzzy rough feasible, β-fuzzy rough efficient. Some techniques for finding a satisfying solution of the problem (P_1) are described in section 4. An efficient solution for this problem is described in section 5. In section 6, we give a numerical example for the sake of illustration. A conclusion and future work are given in section 7.

2. Fuzzy Rough Set Theory

The notions of fuzzy rough set theory are illustrated through the following definitions. For more details see [10].

Definition 2.1 [19]

Let X be a set and L be a lattice. In particular L could be the closed interval [0,1]. A fuzzy set A in X is characterized by a membership function μ_A : X → L, which associates with each point x ∈ X its degree of membership $\mu_A(x) \in$ L.

Let A and B be fuzzy sets in X. Then,

(1) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \forall x \in X$,

(2) $A \subset B \Leftrightarrow \mu_A(x) \le \mu_B(x) \forall x \in X$,

(3) $C = A \cup B \Leftrightarrow \mu_C(x) = \max[\mu_A(x), \mu_B(x)] \forall x \in X$,

(4) $D = A \cap B \Leftrightarrow \mu_D(x) = \min[\mu_A(x), \mu_B(x)] \forall x \in X$.

Definition 2.2 [8]

The α-cut of a fuzzy set A is

$A_\alpha = \{x \in X : \mu_A(x) \ge \alpha\}$.

Definition 2.3 [8]

A fuzzy set A is said to be convex if for every

$\lambda \in [0,1], x^1, x^2 \in X$ then, $\mu_A((1-\lambda)x^1 + \lambda x^2) \ge \min(\mu_A(x^1), \mu_A(x^2))$.

Definition 2.4 [8]

A fuzzy set A is said to be normalized if there exists $x \in X$ such that $\mu_A(x) = 1$.

Also, it can be easily verified that for any fuzzy subset A of X;

(1) A is convex if and only if A_α is convex.

(2) If A is continuous, then A is convex if and only if A_α is closed interval.

Definition 2.5 [8]

A fuzzy subset $\tilde{A}$ of X which is continuous, convex and normalized is said to be a fuzzy number.

Definition 2.6 [7]

Let X denote the n product $X^1 \times X^2 \times ... \times X^n$ and let f be a function defined by The extension principle states that f can be extend to n-tuples $(a^1, a^2, ..., a^n)$ where A is a fuzzy subset of X as follows;

$$\mu_{f(a^1, a^2, ..., a^n)}(x^1, x^2, ..., x^n) = \sup(\min(\mu_{a^1}(x^1), \mu_{a^2}(x^2), ..., \mu_{a^n}(x^n)))$$

Definition 2.7 [11]

Let U be a nonempty set and let β be a complete sub algebra of the Boolean algebra P(U) of subsets of U. The pair (U,β) is called a rough universe.

Let ζ=(U,β) be a given fixed rough universe. Let R be the relation defined as follows:

$A=(A_L, A_U) \in R$ if and only if $A_L, A_U \in \beta$, $A_L \subset A_U$ elements of R are called rough sets and elements of β are called exact sets. Identify the element $(X,X) \in R$ with the element $X \in \beta$ and hence an exact set is a rough set in the sense of the above identification. But a rough set need not be exact set; for example if U is any nonempty set, then (Φ,β) is a rough set which is not exact.

Let $A=(A_L, A_U)$ and $B=(B_L, B_U)$ be any two rough sets, then;

(1) $A = B \Leftrightarrow A_L = B_L, A_U = B_U$;

(2) $A \subset B \Leftrightarrow A_L \subset B_L, A_U \subset B_U$;

(3) $A \cup B = (A_L \cup B_L, A_U \cup B_U)$;

(4) $A \cap B = (A_L \cap B_L, A_U \cap B_U)$.

Definition 2.8 [10]

Let U be a set and β a Boolean sub algebra of the Boolean algebra of all subsets of U. Let L be a lattice. Let X be a rough set then, $X=(X_L,X_U)\in \beta^2$ with $X_L \subset X_U$.

A fuzzy rough set $A=(A_L, A_U)$ in X is characterized by a pair of maps $\mu_{A_L}: X_L \to L$, $\mu_{A_U}: X_U \to L$ with the property that $\mu_{A_L}(x) \le \mu_{A_U}(x)\ \forall x \in X_U$.

For any two fuzzy rough sets $A = (A_L, A_U), B = (B_L, B_U)$ in X then;

(1) $A = B \Leftrightarrow \mu_{A_L}(x) = \mu_{B_L}(x) \forall x \in X_L, \mu_{A_U}(x) = \mu_{B_U}(x) \forall x \in X_U$;

(2) $A \subset B \Leftrightarrow \mu_{A_L}(x) \le \mu_{B_L}(x) \forall x \in X_L, \mu_{A_U}(x) \le \mu_{B_U}(x) \forall x \in X_U$;

(3) $C = A \cup B \Leftrightarrow \mu_{C_L}(x) = \max\{\mu_{A_L}(x), \mu_{B_L}(x)\} \forall x \in X_L$,
$\mu_{C_U}(x) = \max\{\mu_{A_U}(x), \mu_{B_U}(x)\} \forall x \in X_U$;

(4) $D = A \cap B \Leftrightarrow \mu_{D_L}(x) = \min\{\mu_{A_L}(x), \mu_{B_L}(x)\} \forall x \in X_L$,
$\mu_{D_U}(x) = \min\{\mu_{A_U}(x), \mu_{B_U}(x)\} \forall x \in X_U$.

Definition 2.9

The α-cut of a fuzzy rough set A is $A_\alpha = (A_{L_\alpha}, A_{U_\alpha})$ where,
$A_{L_\alpha} = \{x \in X_L : \mu_{A_L}(x) \ge \alpha\}, A_{U_\alpha} = \{x \in X_U : \mu_{A_U}(x) \ge \alpha\}$.

Definition 2.10

A fuzzy rough set $A = (A_L, A_U)$ is said to be convex if A_L and A_U are convex fuzzy sets.

i.e.

$$\mu_{A_L}((1-\lambda)x^1 + \lambda x^2) \ge \min(\mu_{A_L}(x^1), \mu_{A_L}(x^2)) \forall \lambda \in [0,1], x^1, x^2 \in X_L,$$

$$\mu_{A_U}((1-\lambda)x^1 + \lambda x^2) \ge \min(\mu_{A_U}(x^1), \mu_{A_U}(x^2)) \forall \lambda \in [0,1], x^1, x^2 \in X_U$$

Definition 2.11

A fuzzy rough set $A = (A_L, A_U)$ is said to be normalized if there exist an $x \in X_L$ *such that* $\mu_{A_L}(x) = 1$

Definition 2.12

A fuzzy rough set $A = (A_L, A_U)$ which is continuous, convex and normalized is said to be a fuzzy rough number.

Definition 2.13

Let

$X = (X_L, X_U) = ((X_L^1, X_U^1), (X_L^2, X_U^2), \dots, (X_L^n, X_U^n))$, and f be a function defined by $f:(X_L, X_U) \to (R, R)$. The extension principle states that f can be extended to n-tuples ordered pairs $((a_L^1, a_U^1), (a_L^2, a_U^2), \dots, (a_L^n, a_U^n))$ where $A = (A_L, A_U)$ is a fuzzy rough subset of (X_L, X_U) as follows;

$$\mu_{f((a_L^1,a_U^1),(a_L^2,a_U^2),\dots,(a_L^n,a_U^n))}((x_L^1,x_U^1),(x_L^2,x_U^2),\dots,(x_L^n,x_U^n)) = (\sup(\min(\mu_{a_L^1}(x_L^1), \mu_{a_L^2}(x_L^2), \dots, \mu_{a_L^n}(x_L^n))), \sup(\min(\mu_{a_U^1}(x_U^1), \mu_{a_U^2}(x_U^2), \dots, \mu_{a_U^n}(x_U^n))))$$

3. Problem Formulation

3.1. Statement of The Problem

Consider the mathematical program

$$(P_1): \max\ (\bar{c}^1 x, \bar{c}^2 x, \dots, \bar{c}^k x),$$
$$\bar{A} x \le \bar{b},$$
$$x \ge 0$$

where $\bar{c}^j (j = 1,2,\dots,k)$ are n-vectors, $\bar{b}$ is an m-vector and $\bar{A}$ an $m \times n$ matrix, all having fuzzy rough components.

By using the concept of fuzzy rough sets, problem (P_1) converts to the following two fuzzy problems. The first of the lower bound for the rough parameters and the second of the upper bound for the rough parameters.

$$(P_L): \max(\bar{c}_L^1 x, \bar{c}_L^2 x, \dots, \bar{c}_L^k x),$$
$$\bar{A}_L x \le \bar{b}_L$$
$$x \ge 0.$$

and

$$(P_U): \max(\bar{c}_U^1 x, \bar{c}_U^2 x, \dots, \bar{c}_U^k x),$$
$$\bar{A}_U x \le \bar{b}_U$$
$$x \ge 0.$$

Here, we must specify the appropriate solution concept for this problem. For this task, we define the following solution concepts.

3.2. α-Fuzzy Rough Feasible, β-Fuzzy Rough Efficient and satisfying solution for (p_1).

Definition 3.2.1 [α-Fuzzy Rough Feasible]

Consider $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m), \alpha_i \in [0,1]$.

$x \in X = \{x \in R^n : x \geq 0\}$ is said to be α-fuzzy rough feasible for (P_1) if

$\mu_{\tilde{F}_{i_L}}(F_{i_L}) \geq \alpha_i, \mu_{\tilde{F}_{i_U}}(F_{i_U}) \geq \alpha_i$.

where,

$\widetilde{F}_{i_L} = \{\widetilde{A}_{i_L} x \leq \widetilde{b}_{i_L}, x \geq 0\}$,

$\widetilde{F}_{i_U} = \{\widetilde{A}_{i_U} x \leq \widetilde{b}_{i_U}, x \geq 0\}, i = 1,2,...,m.$

Using the extension principle we have

$$\mu_{\tilde{F}_{i_L}} = \sup\min(\mu_{\tilde{a}_{i1_L}}(a_{i1_L}), \mu_{\tilde{a}_{i2_L}}(a_{i2_L}), ..., \mu_{\tilde{a}_{in_L}}(a_{in_L}), \mu_{\tilde{b}_{i_L}}(b_{i_L}))$$

$$\mu_{\tilde{F}_{i_U}} = \sup\min(\mu_{\tilde{a}_{i1_U}}(a_{i1_U}), \mu_{\tilde{a}_{i2_U}}(a_{i2_U}), ..., \mu_{\tilde{a}_{in_U}}(a_{in_U}), \mu_{\tilde{b}_{i_U}}(b_{i_U}))$$

Consider now the following mathematical program;

$$(P_2): \max\ (\bar{c}^1 x, \bar{c}^2 x, ..., \bar{c}^k x),$$
$$x \in D \subset X = \{x \in R^n : x \geq 0\}$$

Definition 3.2.2 [β-Fuzzy Rough Efficient]

$x^0 \in D$ is β-fuzzy rough efficient for (P_2); if there is no $x \in D, l \in (1,2,...,k) : \mu_{\tilde{C}_L}(C_L) \geq \beta, \mu_{\tilde{C}_U}(C_U) \geq \beta$

where,

$$C_L = \{(c_L^1, ..., c_L^k) \in R^{kn} : c_L^1 x \geq c_L^1 x^0, ..., c_L^{l-1} x \geq c_L^{l-1} x^0, c_L^l x > c_L^l x^0, c_L^{l+1} x \geq c_L^{l+1} x^0, ..., c_L^k x \geq c_L^k x^0\}.$$

$$C_U = \{(c_U^1, ..., c_U^k) \in R^{kn} : c_U^1 x \geq c_U^1 x^0, ..., c_U^{l-1} x \geq c_U^{l-1} x^0, c_U^l x > c_U^l x^0, c_U^{l+1} x \geq c_U^{l+1} x^0, ..., c_U^k x \geq c_U^k x^0\}.$$

$\tilde{C}_L, \tilde{C}_U$ are the set of all lower and upper fuzzy coefficient for the fuzzy rough coefficient $\widetilde{C}$ to problem (P_1).

On account of the extension principle,

$$\mu_{\tilde{C}_L}(c_L) = \sup\min(\mu_{\bar{c}_L^1}(c_L^1), \mu_{\bar{c}_L^2}(c_L^2), ..., \mu_{\bar{c}_L^k}(c_L^k)) \geq \beta,$$
$$\mu_{\tilde{C}_U}(c_U) = \sup\min(\mu_{\bar{c}_U^1}(c_U^1), \mu_{\bar{c}_U^2}(c_U^2), ..., \mu_{\bar{c}_U^k}(c_U^k)) \geq \beta$$

Definition 3.2.3

$x^0 \in X$ is an (α,β) satisfying solution for (p_1) if and only if x^0 is β-fuzzy rough efficient for the program

$$(P_3): \max\ (\bar{c}^1 x, \bar{c}^2 x, ..., \bar{c}^k x),$$
$$x \in X^\alpha$$

where X^α denotes the set of all α-fuzzy rough feasible actions for (P_1).

4. Characterization of An (α,β)-Satisfying Solution for (p_1).

Consider the mathematical program

$$(P_4): \max\ ((\bar{c}^1)_\beta x, (\bar{c}^2)_\beta x, ..., (\bar{c}^k)_\beta x).$$
$$x \in X^\alpha$$

where $(\bar{c}^i)_\beta = ((\bar{c}_1^i)_\beta (\bar{c}_2^i)_\beta ... (\bar{c}_n^i)_\beta)$ and $(\bar{c}_j^i)_\beta$ denotes the β-cut of the fuzzy rough number $\bar{c}_j^i$.

By the convexity assumption, $(\widetilde{c}_j^i)_\beta, j = 1,2,...,n, i = 1,2,...,k$ are rough real intervals that will be denoted as $[C_j^{is}(\lambda), C_j^{im}(\theta)]$ and given by $[(1-\lambda)c_{j_U}^{is} + \lambda c_{j_L}^{is}, (1-\theta)c_{j_L}^{im} + \theta c_{j_U}^{im}], 0 \leq \lambda, \theta \leq 1$

Let now $\phi_\beta(\lambda, \theta)$ be the set of all $k \times n$ rough real matrices $C(\lambda, \theta) = (c_{ij}(\lambda, \theta))$ with,

$$c_{ij}(\lambda, \theta) \in [(1-\lambda)\bar{c}_{j_U}^{is} + \lambda\bar{c}_{j_L}^{is}, (1-\theta)\bar{c}_{j_L}^{im} + \theta\bar{c}_{j_U}^{im}], 0 \leq \lambda, \theta \leq 1.$$

It is clear that (P_4) may be written as:

$$\max\ (cx : x \in X^\alpha, c \in \phi_\beta(\lambda, \theta)).$$

(P_4) is then a family of pair multiple objective linear programs.

Definition 4.1

x^0 is rough efficient for (P_4) if and only if there is no $c(\lambda, \theta) \in \phi_\beta(\lambda, \theta), x \in X^\alpha : cx \geq cx^0$ with at least strict one inequality holds. In other words, x^0 is rough efficient for (P_4) if and only if x^0 is efficient for

$$\max\ \ cx \quad \forall\, c(\lambda, \theta) \in \phi_\beta(\lambda, \theta),$$
$$x \in X^\alpha$$

Now we are going to give the following result.

Theorem 4.1

x^0 is an (α,β)-satisfying solution for (P_1) if and only if x^0 is an rough efficient solution for (P_4).

Proof.

Suppose that x^0 is an (α,β) -satisfying solution for (P_1) then by definition, x^0 is α-fuzzy rough feasible and β- fuzzy rough efficient for (P_3). Assume now that x^0 is not efficient for (P_4). Then there is

$x^1 \in X^{\alpha}, (c^1, c^2, \ldots, c^k)$ $with c^i \in (\bar{c}^i)_\beta$; and

$\bar{c}^i x^1 \geq \bar{c}^i x^0 \forall i \in \{1,2,\ldots,k\}$

$i_0 \in \{1,2,\ldots,k\}$ such that

$$c^{i_0} x^1 > c^{i_0} x^0 \qquad (1)$$

As $c^i \in (\bar{c}^i)_\beta, i = 1,2,\ldots,k$ we have also

$$\min(\mu_{\bar{c}^1_L}(c^1), \mu_{\bar{c}^2_L}(c^2), \ldots, \mu_{\bar{c}^k_L}(c^k)) \geq \beta \text{ and}$$

$$\min(\mu_{\bar{c}^1_U}(c^1), \mu_{\bar{c}^2_U}(c^2), \ldots, \mu_{\bar{c}^k_U}(c^k)) \geq \beta$$

(2)

By (1),(2) we have

$$\sup_{(c^1,c^2,\ldots,c^k) \in T^L_{i_0}} \min(\mu_{\bar{c}^1_L}(c^1), \mu_{\bar{c}^2_L}(c^2), \ldots, \mu_{\bar{c}^k_L}(c^k)) = \mu_{\bar{c}L}(c^1x^1 \geq c^1x^0, \ldots, c^{i_0-1}x^1 \geq c^{i_0-1}x^0, c^{i_0}x^1 > c^{i_0}x^0, c^{i_0+1}x^1 \geq c^{i_0+1}x^0, \ldots, c^k x^1 \geq c^k x^0) \geq \beta,$$

$$\sup_{(c^1,c^2,\ldots,c^k) \in T^U_{i_0}} \min(\mu_{\bar{c}^1_U}(c^1), \mu_{\bar{c}^2_U}(c^2), \ldots, \mu_{\bar{c}^k_U}(c^k)) = \mu_{\bar{c}U}(c^1x^1 \geq c^1x^0, \ldots, c^{i_0-1}x^1 \geq c^{i_0-1}x^0, c^{i_0}x^1 > c^{i_0}x^0, c^{i_0+1}x^1 \geq c^{i_0+1}x^0, \ldots, c^k x^1 \geq c^k x^0) \geq \beta$$

where.

$$T^t_{i_0} = \{(c^1, \ldots, c^k) \in R^{nk} \mid c^1x^1 \geq c^1x^0, \ldots, c^{i_0-1}x^1 \geq c^{i_0-1}x^0, c^{i_0}x^1 > c^{i_0}x^0, c^{i_0+1}x^1 \geq c^{i_0+1}x^0, \ldots, c^k x^1 \geq c^k x^0\}, t = L, U.$$

This contradicts the β-rough efficiency of x^0 for (P_3) and the if part of the theorem is established.

To show the only if part, suppose x^0 is efficient for (P_4) and not (α,β) -satisfying solution for (P_1). Then there is $x^2 \in X^\alpha; S \in \{1,2,\ldots,k\}$ such that

$\mu_{\bar{c}_L}(c_L) \geq \beta, \mu_{\bar{c}_U}(c_U) \geq \beta$

where

$$\mu_{\bar{c}^t}(c^1x^2 \geq c^1x^0, \ldots, c^{s-1}x^2 \geq c^{s-1}x^0, c^s x^2 > c^s x^0, c^{s+1}x^2 \geq c^{s+1}x^0, \ldots, c^k x^2 \geq c^k x^0) \geq \beta, t = L, U$$

i.e.

$$\sup_{(c^1,c^2,\ldots,c^k) \in T^t_s} \min(\mu_{\bar{c}^1}(c^1), \ldots, \mu_{\bar{c}^k}(c^k)) \geq \beta.$$

where,

$$T^t_s = \{(c^1, \ldots, c^k) \in R^{nk} \mid c^1x^2 \geq c^1x^0, \ldots, c^{s-1}x^2 \geq c^{s-1}x^0, c^s x^2 > c^s x^0, c^{s+1}x^2 \geq c^{s+1}x^0, \ldots, c^k x^2 \geq c^k x^0\}, t = L, U$$

By using the extension principle then, we have

$$\text{Sup} \min(\mu_{\bar{c}^1_L}(c^1_L), \mu_{\bar{c}^2_L}(c^2_L), \ldots, \mu_{\bar{c}^k_L}(c^k_L)) \geq \beta,$$

and

(3)

$$\text{Sup} \min(\mu_{\bar{c}^1_U}(c^1_U), \mu_{\bar{c}^2_U}(c^2_U), \ldots, \mu_{\bar{c}^k_U}(c^k_U)) \geq \beta.$$

For this supremum to exist there is $(p^1, p^2, \ldots, p^k)$ satisfying the following constraints;

$$p^1x^2 \geq p^1x^0, p^2x^2 \geq p^2x^0, \ldots, p^{s-1}x^2 \geq p^{s-1}x^0, p^s x^2 > p^s x^0, p^{s+1}x^2 \geq p^{s+1}x^0, \ldots, p^k x^2 \geq p^k x^0.$$

(4)

Suppose now that for all $(p^1, p^2, \ldots, p^k)$ satisfying the system (4). we have min $(\mu_{\bar{c}^1_L}(p^1), \mu_{\bar{c}^2_L}(p^2), \ldots, \mu_{\bar{c}^k_L}(p^k)) < \beta$, and

min

$(\mu_{\bar{c}^1_U}(p^1), \mu_{\bar{c}^2_U}(p^2), \ldots, \mu_{\bar{c}^k_U}(p^k)) < \beta$, then,

Sup min

$$(\mu_{\bar{c}^1_L}(p^1), \mu_{\bar{c}^2_L}(p^2), \ldots, \mu_{\bar{c}^{s-1}_L}(p^{s-1}), \mu_{\bar{c}^s_L}(p^s), \mu_{\bar{c}^{s+1}_L}(p^{s+1}), \ldots, \mu_{\bar{c}^k_L}(p^k)) < \beta.$$

Sup min

$$(\mu_{\bar{c}^1_U}(p^1), \mu_{\bar{c}^2_U}(p^2), \ldots, \mu_{\bar{c}^{s-1}_U}(p^{s-1}), \mu_{\bar{c}^s_U}(p^s), \mu_{\bar{c}^{s+1}_U}(p^{s+1}), \ldots, \mu_{\bar{c}^k_U}(p^k)) < \beta.$$

Contradicting (3). There is then $(p^1, p^2, \ldots, p^k)$ satisfying (4) such that $\min(\mu_{\bar{c}^1_L}(P^1), \ldots, \mu_{\bar{c}^k_L}(P^K)) \geq \beta$,

$$\min(\mu_{\bar{c}^1_U}(P^1), \ldots, \mu_{\bar{c}^k_U}(P^K)) \geq \beta.$$

(5)

By (5), $\mu_{\bar{c}^i}(p^i) \geq \beta, i = 1,2,\ldots,k$

i.e. $p^i \in (c^i)_\beta, i = 1,2,\ldots,k$

(6)

(4) and (6) contradict the efficiency of x^0 for (p_4) and we have done.

5. A Solution for (p_4)

The following notations will facilitate further discussions.

$M_\beta(\lambda,\theta)$ denotes the subset of $\phi_\beta(\lambda,\theta)$ composed by matrices $C(\lambda,\theta)$ having elements of each column at the upper bound or at the lower bound.

i.e. if $C(\lambda,\theta) \in M_\beta(\lambda,\theta)$ then either

$$C_j^r(\lambda)=\begin{pmatrix} c_j^{r1}(\lambda) \\ c_j^{r2}(\lambda) \\ \vdots \\ c_j^{rn}(\lambda) \end{pmatrix} \text{ or } C_j^m(O)=\begin{pmatrix} c_j^{m1}(\theta) \\ c_j^{m2}(\theta) \\ \vdots \\ c_j^{mn}(O) \end{pmatrix}$$

where $c_j^{ri}(\lambda)$ and $c_j^{mi}(\theta)$ are the left and the right end point of $(\widetilde{C}_j^i(\lambda,O))_\beta$ respectively.

Lemma 5.1

A necessary and sufficient condition for x^0 to be efficient for the multiple objective program max $Cx, x\in X$

Is that there is $\lambda>0$ such that x^0 solves the mathematical program max $\lambda Cx, x\in X$.

A fascinating point is that the following program yields an (α,β)-satisfying solution for (P_1):

(P_5): max $q^0x, x\in X^a$

where q^0 is a solution of the system

$V'C'(\lambda,O)-q=0$ $\forall i$ such that

$C'(\lambda,O)\in M_\beta(\lambda,O)$, (7)

Proposition 5.1

If x^0 is optimal for (P_5) then x^0 is efficient for (P_4).

Proof.

As q^0 is a solution of (7), $\forall i$ such that $C'(\lambda,O)\in M_\beta(\lambda,O)$, there is $V'\in R^k, V'>0$ such that $V'C'(\lambda,O)=q^0$.

i.e. $\forall i$ such that $C'(\lambda,O)\in M_\beta(\lambda,O)$, x^0 solves $\max(V'C'(\lambda,O)x : x\in X^a)$.

By lemma 5.1., x^0 is efficient for max $(C'(\lambda,O)x : x\in X^a)$ $\forall C'(\lambda,O)\in\phi_\beta(\lambda,O)$.

i.e. x^0 is efficient for (P_4) as desired.

Corollary 5.1

If x^0 is optimal for (P_5) then x^0 is an (α,β)-satisfying for (P_1).

This statement follows directly from Proposition 5.1. and theorem 4.1.

6. A Numerical Example

In this section we are going to give a numerical example.

Consider the following multiple objective linear fuzzy rough program

(P_6) max $(\bar{c}^1x, \bar{c}^2x)$

such that

$$\begin{aligned} &x_1+x_2\ge 250,\\ &x_1\le 200,\\ &2x_2\le 200,\\ &2x_1+1.5x_2\le 480,\\ &3x_1+4x_2\le 900; x_1,x_2\ge 0. \end{aligned}$$

where $\bar{c}^1=(\bar{c}_{11},\bar{c}_{12}), \bar{c}^2=(\bar{c}_{21},\bar{c}_{22})$ and $\bar{c}_{ij}=(\bar{c}_{ij_L},\bar{c}_{ij_U})$ are fuzzy rough numbers characterized by the membership functions shown in Figure (1a-d).

This problem can be converted into the following two fuzzy problems. The first problem is,

(P_{6_L}) max $(\bar{c}_L{}^1x, \bar{c}_L{}^2x)$

such that

$$\begin{aligned} &x_1+x_2\ge 250,\\ &x_1\le 200,\\ &2x_2\le 200,\\ &2x_1+1.5x_2\le 480,\\ &3x_1+4x_2\le 900; x_1,x_2\ge 0. \end{aligned}$$

where, $\bar{c}_L{}^1=(\bar{c}_{11_L},\bar{c}_{12_L}), \bar{c}_L{}^2=(\bar{c}_{21_L},\bar{c}_{22_L})$ and $\bar{c}_{ij_L}$ are fuzzy numbers characterized by the membership functions shown in Figure (2a-d).

The second problem is,

(P_{6_U}): max $(\bar{c}_U{}^1x, \bar{c}_U{}^2x)$

such that

$$\begin{aligned} &x_1+x_2\ge 250,\\ &x_1\le 200,\\ &2x_2\le 200,\\ &2x_1+1.5x_2\le 480,\\ &3x_1+4x_2\le 900; x_1,x_2\ge 0. \end{aligned}$$

where, $\bar{c}_U{}^1=(\bar{c}_{11_U},\bar{c}_{12_U}), \bar{c}_U{}^2=(\bar{c}_{21_U},\bar{c}_{22_U})$ and $\bar{c}_{ij_U}$ are fuzzy numbers characterized by the membership functions shown in Figure (3a-d).

In (P_6), let $\beta=0$ then we obtain

$(\bar{c}_{11})_\beta = [0(1-\lambda)+1\lambda, 2(1-\theta)+3\theta] = [\lambda, \theta+2]$,

$(\bar{c}_{12})_\beta = [0(1-\lambda)+1\lambda, 1(1-\theta)+2\theta] = [\lambda, \theta+1]$,

$(\bar{c}_{21})_\beta = [-2(1-\lambda)+0\lambda, 0(1-\theta)+2\theta] = [2\lambda-2, 2\theta]$,

$(\bar{c}_{22})_\beta = [-3(1-\lambda)+1\lambda, 1(1-\theta)+2\theta] = [4\lambda-3, \theta+1]$.

and $0 \le \lambda, \theta \le 1$

By putting $\lambda = 1$, $\theta = 0$ then we get $(\bar{c}_{11})_\beta = [1,2], (\bar{c}_{12})_\beta = \{1\} = (\bar{c}_{22})_\beta, (\bar{c}_{21})_\beta = \{0\}$.

The subset M_β of ϕ_β is composed of the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $q^0 = \begin{pmatrix} 4.4 \\ 2.9 \end{pmatrix}$ the system

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_1^1 \\ V_2^1 \end{pmatrix} = \begin{pmatrix} 4.4 \\ 2.9 \end{pmatrix}$$

As well as the system

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_1^2 \\ V_2^2 \end{pmatrix} = \begin{pmatrix} 4.4 \\ 2.9 \end{pmatrix}$$

has positive solutions given by

$$\begin{pmatrix} V_1^1 \\ V_2^1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 2.9 \end{pmatrix}, \quad \begin{pmatrix} V_1^2 \\ V_2^2 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 2.9 \end{pmatrix}.$$

respectively.

As constraints are crisp, the set X of points of R^2 satisfying (P_6) is nothing but the set of 1-fuzzy rough actions. By the corollary, the program max $(q^0 x \mid x \in X^\alpha)$ yields a (1,0)-satisfying solution for (P_6).

Solving this linear program, we obtain the solution $(x_1^*, x_2) = (150, 100)$.

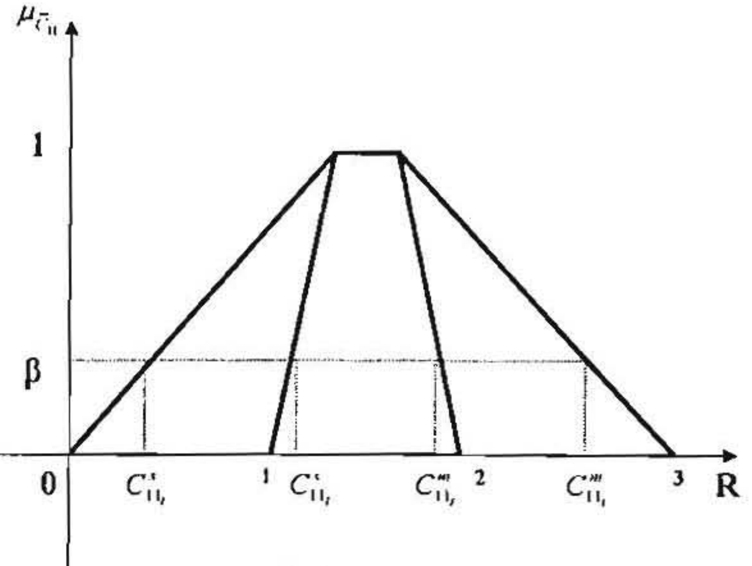

Fig 1-a

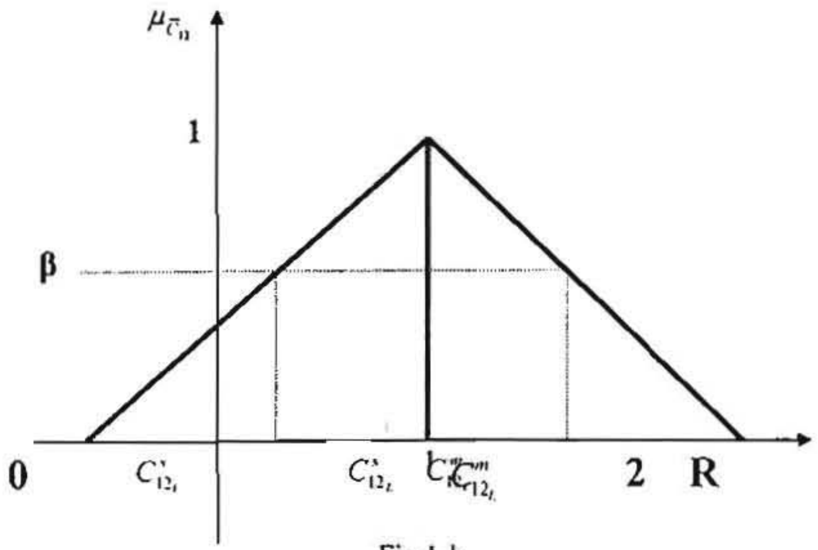

Fig 1-b

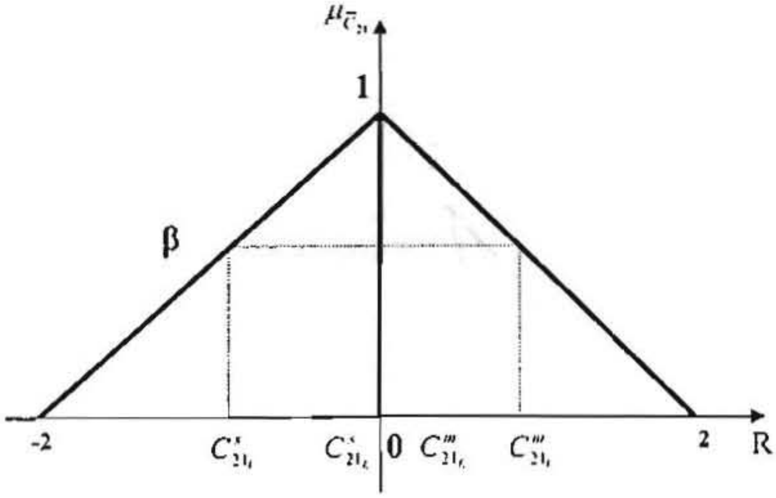

Fig 1-c

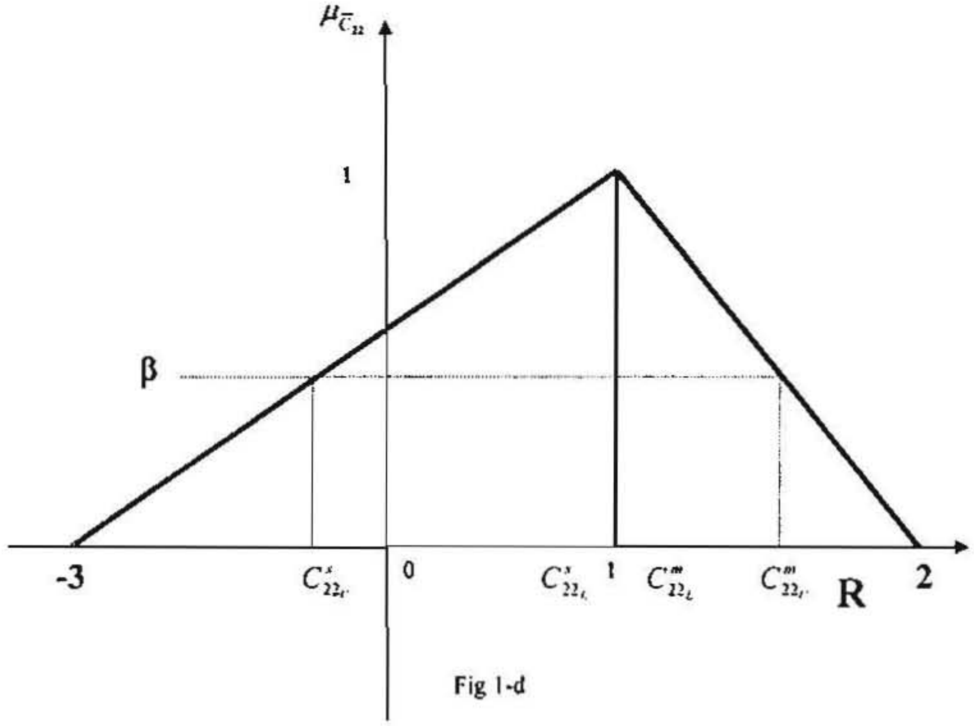

Fig 1-d

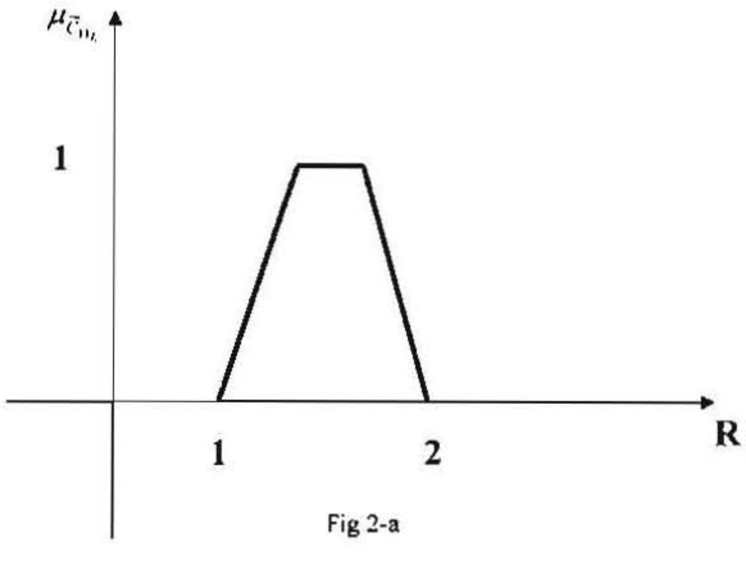

Fig 2-a

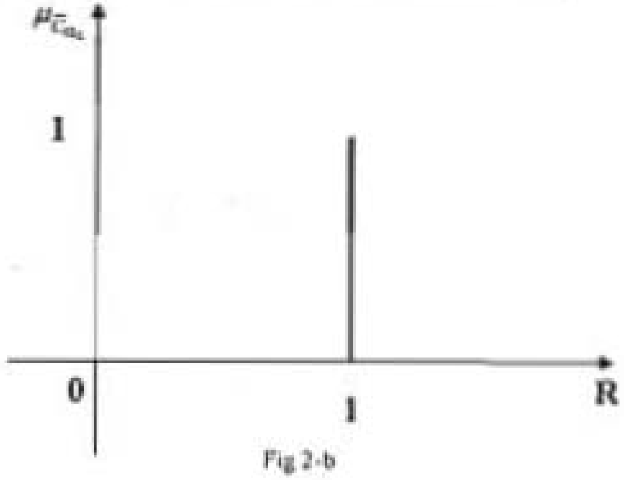

Fig 2-b

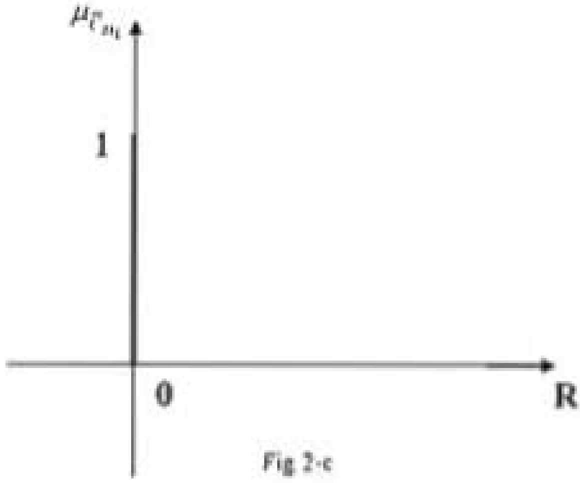

Fig 2-c

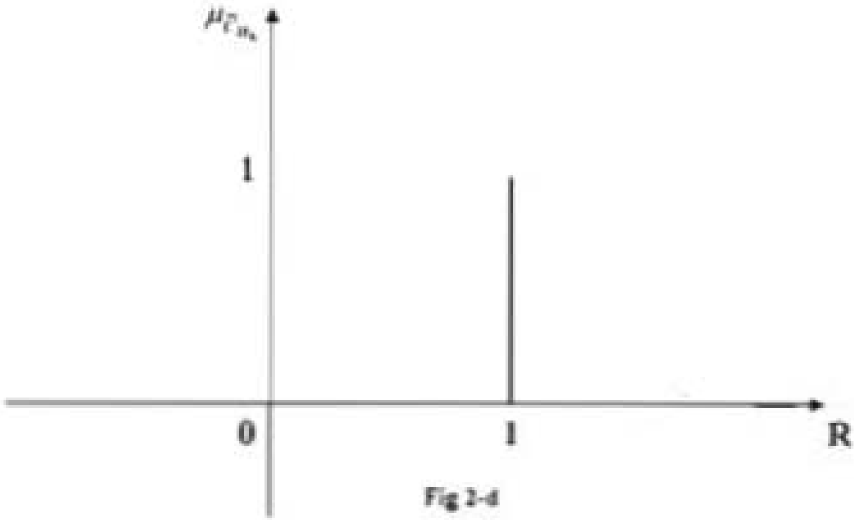

Fig 2-d

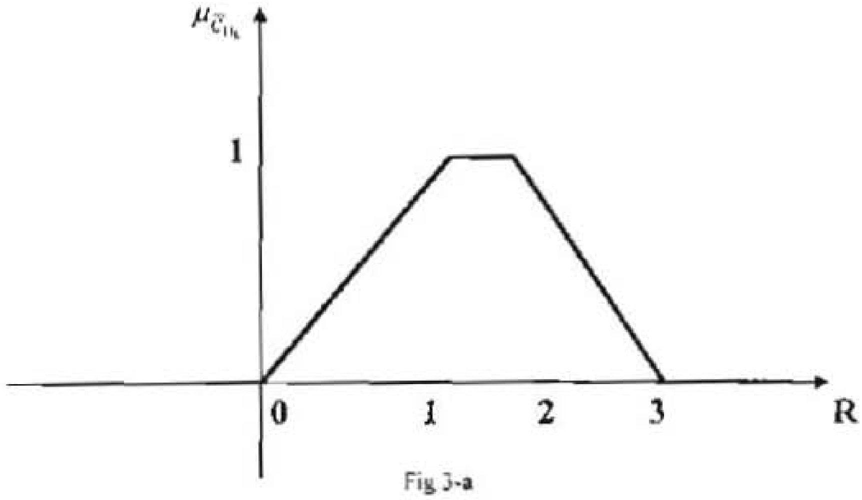

Fig 3-a

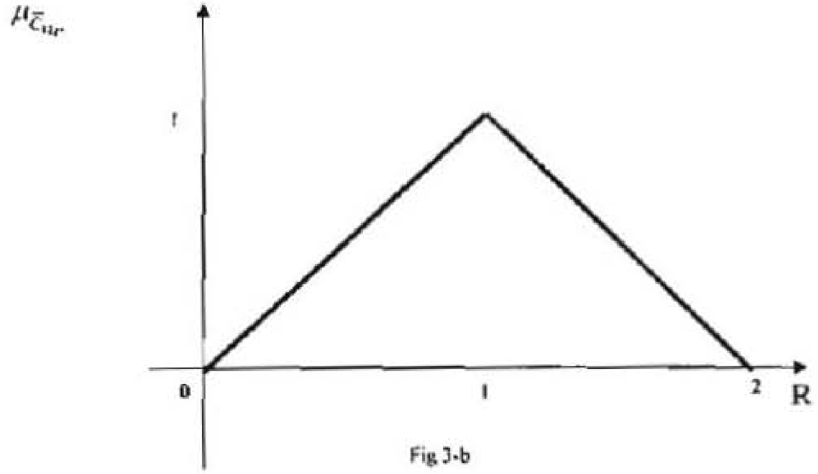

Fig 3-b

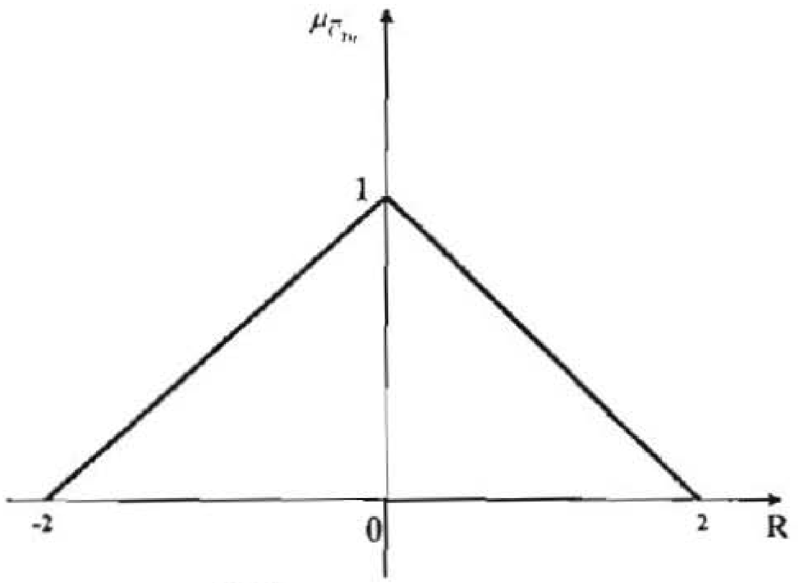

Fig 3-c

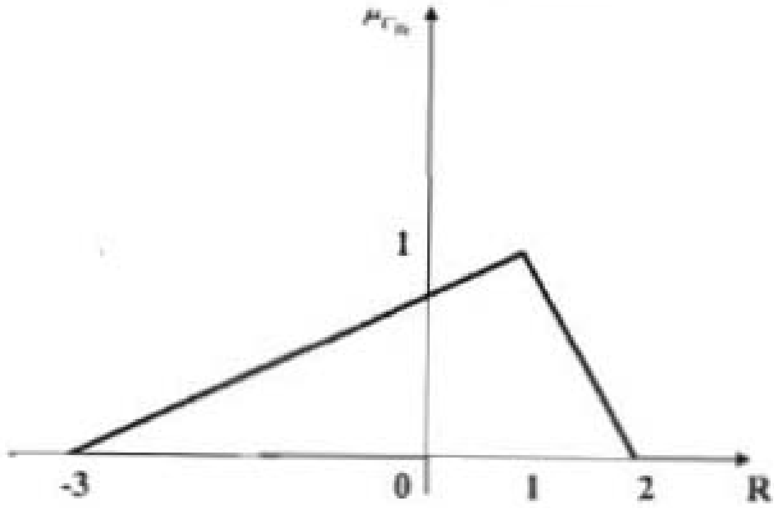

Fig 3-d

7. Conclusion

There are a large number of real world problem that can be cast in form of multiple objective fuzzy rough programming problems. A significant outcome of this paper is the investigation of the problem by incorporating fuzzy rough numbers into a multiple objective linear programming framework. Inspired by the techniques of Luhandjula in [7], we characterized the solution for this problem via the concepts of α-fuzzy rough feasibility and β-fuzzy rough efficiency. A numerical example is given for the sake of development theory.

The extension of the current work to real life problems and parametric studies is now under active consideration. This will be the subject of the next paper.

References

[1] F. F. Baldwin, N. C. F. Guild, Comparison of fuzzy sets on the same decision space, Fuzzy Sets and Systems Vol. 2, pp. 213-231, 1979.

[2] R. Bellman and L. A. Zadeh, Decision making in a fuzzy environment. Management Sci. Vol. 17, pp. 141-164, 1970.

[3] D. Dubois, H. Prade, Ranking fuzzy numbers in the setting of possibility theory, Inform. Sci. Vol. 30, pp. 183-224, 1983.

[4] P. Fortemps and M. Roubens, Ranking and defuzzification methods based on area compensation, Fuzzy Sets and Systems Vol. 82,pp. 319-330, 1996.

[5] T. B. Iwinski, Algebraic approach to rough sets, Bull. Polish Acad. Sci. Math. Vol. 35, pp. 673-683, 1987.

[6] H. W. Kuhn and A. W. Tucker, Nonlinear programming, Proc. 2nd Berkeley Sympos. On Mathematics, statistics and Probability, University of California Press, Berkeley, pp.481-492, 1951.

[7] M.K.Luhandjula, On multiple objective programming with possibilistic coefficients, Fuzzy sets and systems Vol. 21, pp. 135-145, 1987.

[8] H. R. Maleki, M. Tata and M. Mashinchi, Linear programming with fuzzy variables. Fuzzy Sets and Systems Vol. 109, pp. 21-33, 2000.

[9] K. Nakamura, Preference relations on a set of fuzzy utilities, Fuzzy Sets and Systems Vol. 20,pp. 147-162, 1986.

[10] S. Nanda and S. Majumdar, Fuzzy rough sets, Fuzzy Sets and Systems Vol. 45,pp. 157-160, 1992.

[11] Z. Pawlak, Rough sets, Internat J. Inform. Comput. Sci. Vol. 11(5), pp. 341-356, 1982.

[12] Z. Pawlak, Rough sets and fuzzy sets, Fuzzy Sets and Systems Vol. 17,pp. 99-102, 1985.

[13] J. Ramik and J. Rimanek, Inequality relation between fuzzy numbers and its use in fuzzy optimization, Fuzzy Sets and Systems Vol. 16,pp. 123-138, 1985.

[14] M. Roubens, Comparison of flat fuzzy numbers, in : N. Badler, A. Kandel (Eds.), Proc. NAFIPS Vol. 86, pp. 462-476, 1986.

[15] K. Tamura and S. Miura, On linear vector maximization problems, J. Oper. Res. Sci. Japan Vol. 20, pp. 139-149, 1977.

[16] H. Tanaka and K. Asai, Fuzzy linear programming based on fuzzy functions, Bull. Univ. Osaka Perfect. Ser. A Vol. 29, pp.113-125, 1980.

[17] J. Vangeldere, Pareto weak minimality and various connected properties, in W. Oettli and F. Steffans (Eds.), Proc. Third Sympos. On Operations Research, Band 31, University of Mannheim, Mannheim pp. 663-676, 1978.

[18] L. A. Zadeh, Optimality and nonscalar-valued performance criteria, IEEE Trans. Automatic control, A C-8 ,pp. 59-61, 1963. [19] L. A. Zadeh, Fuzzy sets, Information and Control Vol. 8, pp. 338-353, 1965.