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COMPUTATIONAL METHOD FOR SOLVING THE REGULARISED LONG WAVE EQUATION

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ABSTRACT

A finite element solution of the regularised long wave equation based on collocation method using quintic splines as element shape functions, is set up. A linear stability analysis shows the scheme to be unconditionally stable. Test problems, including the migration and interaction of solitary waves, are used to validate the method which is found to be accurate and efficient. The three invariants of the motion are evaluated to determine the conservation properties of the algorithm. The temporal evaluation of a Maxwellian initial pulse is then studied.

INTRODUCTION

The regularised long wave equation (**RLW**) is an important nonlinear wave equation. Solitary waves are wave packets or pulses which propagate in nonlinear dispersive media. The dynamical balance between the nonlinear and dispersive effects of these waves retain a stable wave form. A soliton is a very special type of solitary wave which also keeps its wave form after collision with other solitons.

The regularised long wave (**RLW**) equation is an alternative description of nonlinear dispersive waves to the more usual Korteweg-de Vries (**KDV**) equation (Peregrine). It has been shown to have solitary wave solutions and to govern a large number of important physical phenomena such as shallow water waves and plasma wave (Peregrine and Abdulloev et al.).

Few analytic solutions are known. Approximate solutions based on finite difference techniques (Eilbeck and McGuire), Range Kutta and predictor corrector methods (Bona et al.) and Galerkin's method (Alexander and Morris), are also well known. Wahlbin using a trial function composed of hermite cubic polynomials, while Alexander and Morris constructed a global trial function mainly from cubic splines. In the latter case the closure at the boundaries affected with quintic polynomials and an implicit finite element approach is used, in which the element matrices were not explicitly formed, but the global trial function was used directly to determine the global equations. Alexander and Morris solved the resulting system of ordinary differential equations using the **IMSL** Library (1975) routine **DREBS**. Recently Gardner sets up an implicit finite element solution using cubic splines (Hearn) as the element "shape" and weight functions throughout the solution region, and the Galerkin's method. In the present paper we set up explicit finite element solution using quinitic splines as the element "shape" and weight functions throughout the solution region. The element matrices are determined algebraically, and the equations governing the problems are obtained by explicitly assembling together the element matrices to obtain the full global matrix equation. The time integration used to solve the resulting system of ordinary differential equations involves a Crank-Nicolson scheme together with an inner interaction to cope with the nonlinear term and details of this method is given in section 2. A linear stability analysis of the numerical scheme shows that it is unconditionally stable. The finite element method is shown to represent accurately the migration of a solitary wave. Finally the evaluation of a Maxwellian initial condition into stable solitary waves is investigated.

THE GOVERNING EQUATION

The RLW equation for the long waves propagating in the positive x-direction has the form (Peregrine):

$$V_t + V_x + V V_x - \nu V_{xxt} = 0,$$

where v is a positive parameter and the subscripts x and t denote the differentiation with respect to x and t respectively, with the physical boundary condition $V \longrightarrow 0$ as $x \longrightarrow \pm\infty$.

Using the mapping $U = V + 1$ we can transform this equation to

$$U_t + UU_x - vU_{xxt} = 0 \tag{1}$$

with boundary condition $U \longrightarrow 1$ as $x \longrightarrow \pm\infty$. In this paper we consider the **RLW** equation to be of the form (1) and use the periodic boundary conditions for a region $a \leq x \leq b$. The form of the initial pulse is chosen so that at large distances from the pulse the function U tends to 1 to agree with the physical boundary condition. The region is partitioned into N finite elements of equal length h by the knots x_i such that $a=x_0<x_1<\ldots<x_n=b$. The quintic splines ϕ_i with knots at x_i form a complete basis for the functions defined over [a,b]. A global approximation $U_N(x,t)$ to the solution U(x,t) is given by

$$U_N(x,t) = \sum_{i=-2}^{N+2} \delta_i(t)\, \phi_i(x) \tag{2}$$

where the δ_i's are the time dependent quantities to be determined. Each quintic spline spans 5 finite elements, so that 5 splines cover each element. The spline $\phi_i(x)$ and its 2 principal derivatives vanish outside the region $[x_{i-3}, x_{i+3}]$. In Table 1 the values of ϕ_i and its principal derivatives at the relevent knots are listed. At the knots x_i the numerical solution $U_N(x,t)$ is given by

$$\left.\begin{aligned} U_i &= \delta_{i-2} + 26\delta_{i-1} + 66\delta_i + 26\delta_{i+1} + \delta_{i+2} \\ hU_i' &= 5\,\delta_{i+2} + 50\,\delta_{i+1} - 50\,\delta_{i-1} - 5\,\delta_{i-2} \\ h^2U_i'' &= 20(\delta_{i-2} + 2\delta_{i-1} - 6\delta_i + 2\delta_{i+1} + \delta_{i+2}) \end{aligned}\right\} \tag{3}$$

The function U and its first two derivatives are continuous across element boundaries. We substitute (2) into (1), identify the collocation points with the knots and use equation (3) to evaluate U_i and its space derivatives (Prenter).

Table (1): The quintic spline ϕ

x	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
ϕ_i	0	1	26	66	26	1	0
$h\phi'_i$	0	5	50	0	-50	-5	0
$h^2\phi''_i$	0	20	40	-120	40	20	0

Thus implementing the method of lines leads to a set of ordinary differential equations with the form

$$\dot{\delta}_{i-2} + 26\dot{\delta}_{i-1} + 66\dot{\delta}_i + 26\dot{\delta}_{i+1} + \dot{\delta}_{i+2} +$$
$$\frac{5}{h}Z_i(\delta_{i+2}+10\delta_{i+1}-10\delta_{i-1}-\delta_{i-2}) - \frac{20}{h^2}v(\dot{\delta}_{i-2}+2\dot{\delta}_{i-1}-6\dot{\delta}_i+2\dot{\delta}_{i+1}+\dot{\delta}_{i+2}) = 0,$$
$$i = 0, 1, ..., \qquad (4)$$

Where : $Z_i = \delta_{i-2} + 26\delta_{i-1} + 66\delta_i + 26\delta_{i+1} + \delta_{i+2}$

The system of ordinary differential equations (4) may now be solved using an appropriate software package, for example, by using the routine D02CAF of the Numerical Algorithms Group program library.

In an alternative approach, which is used in this paper, a recurrance relationship based on a Crank-Nicolson approximation in time is derived. Suppose that $\mathbf{d} = (\delta_{-2}, \delta_{-1}, \delta_0, ..., \delta_{N+2})^T$, if the vector of nodal parameters, is linearly interpolated between two time levels n and n+1, then $\mathbf{d}$ and its time derivative are given by

$$\mathbf{d} = \frac{1}{2}(\mathbf{d}^{n+1} + \mathbf{d}^n), \quad \dot{\mathbf{d}} = \frac{1}{\Delta t}(\mathbf{d}^{n+1} - \mathbf{d}^n), \qquad (5)$$

where $\mathbf{d}^n$ are the parameters at the time $n\Delta t$. Hence using Eq.(5) in (4), we have for each knot an equation relating parameters at adjacent time levels, δ_i^{n+1} to δ_i^n

$$\alpha_{i1}\delta_{i-2}^{n+1} + \alpha_{i2}\delta_{i-1}^{n+1} + \alpha_{i3}\delta_i^{n+1} + \alpha_{i4}\delta_{i+1}^{n+1} + \alpha_{i5}\delta_{i+2}^{n+1} =$$

$$\alpha_{i5}\delta_{i-2}^{n} + \alpha_{i4}\delta_{i-1}^{n} + \alpha_{i3}\delta_i^{n} + \alpha_{i2}\delta_{i+1}^{n} + \alpha_{i1}\delta_{i+2}^{n},$$

$$i = 0, 1, 2, ..., N, \qquad (6)$$

where : $\alpha_{i1} = 1\text{-}R_1Z_i\text{-}R_2$, $\quad \alpha_{i2} = 26\text{-}10R_1Z_i\text{-}2R_2$,

$\alpha_{i3} = 66+6R_2$, $\quad \alpha_{i4} = 26+10R_1Z_i\text{-}2R_2$, $\quad \alpha_{i5} = 1+R_1Z_i\text{-}R_2$.

$Z_i = \delta_{i-2} + 26\delta_{i-1} + 66\delta_i + 26\delta_{i+1} + \delta_{i+2}$, $R_1 = \frac{5\Delta t}{2h}$ and $R_2 = \frac{20\nu}{h^2}$.

The system (6) consists of N+1 nonlinear equations in N+5 unknowns $(\delta_{-2}, \delta_{-1}, \delta_0, ..., \delta_{N+2})^T$. To obtain a solution to this system we need 4 additional constraints. These are obtained from the boundary conditions, and can be used to eliminate δ_{-2}, δ_{-1}, δ_{N+1}, δ_{N+2} from the set (6) which then becomes a matrix equation for the N+1 unknowns $\mathbf{d}^{n+1} = (\delta_0, \delta_1, \delta_2, ..., \delta_N)^T$.

$$A(\mathbf{d}^n)\,\mathbf{d}^{n+1} = B(\mathbf{d}^n)\,\mathbf{d}^n + \mathbf{r}, \qquad (7)$$

where $A(\mathbf{d}^n)$ and $B(\mathbf{d}^n)$ are pentadiagonal matrices, and $\mathbf{r}$ is an N+1 vector which depends on the boundary conditions.

The time evolution of the approximate solution $U_N(x,t)$ is determined by the time evolution of the vector $\mathbf{d}^n$. This is found by repeatedly solving the recurrence relationship once the initial vector $\mathbf{d}^0$ has been computed from the initial conditions. The recurrence relationship (7) is pentadiagonal and a direct algorithm for the rapid solution of the equations is available. However, an inner iteration is also needed, at each time step, to cope with the nonlinear terms. The following solution procedure is followed.

1. At time $t = 0$, for the initial step of the inner iteration we approximate A and B by A^* and B^* calculated from $\mathbf{d}^0$ only and obtain a first approximation to

$\mathbf{d}^1$ from (7). We then iterate, using (7) with matrices A and B calculated from $\mathbf{d}=.5(\mathbf{d}^0 + \mathbf{d}^1)$, for up to 10 times to refine the approximation to $\mathbf{d}^1$.

2. At all other time steps we use for matrices A and B, at the first step of the inner iteration, A^* and B^* derived from $\mathbf{d}^* = \mathbf{d}^n + .5(\mathbf{d}^n - \mathbf{d}^{n-1})$ to obtain a first approximation to $\mathbf{d}^{n+1}$ by solving (7). We then iterate, using (7) with matrices A and B calculated from $\mathbf{d} = .5(\mathbf{d}^n + \mathbf{d}^{n+1})$, two or three times to refine the approximation to $\mathbf{d}^{n+1}$.

STABILITY ANALYSIS

An investigation into the stability of the numerical scheme (6) is based on the von Neumann theory in which the growth factor is typically of Fourier mode, defined as

$$\delta_j^n = \hat{\delta}^n e^{ijkh} \tag{8}$$

where, k is the mode number and h is the element size, is determined for a linearisation of the numerical scheme.

The nonlinear term UU_x of Regularised Long Wave equation is linearised by making the quantity U locally constant which is equivalent to assuming that the corresponding values of δ_j^n are equal to a local constant d. Substituting the fourier mode (8) in equation (6) we obtain

$\hat{\delta}^{n+1} = g\,\hat{\delta}^n$ where the growth factor g has the form

$$g = \frac{a - ib}{a + ib}, \tag{9}$$

where : $a = (1 - R_2)\cos(2kh) + (\beta - R_2)\cos(kh) + 33 + 3R_2$,

$$b = -[R_1^*\sin(2kh) + 10R_1^*\sin(kh)],\ R_1^* = (120d)R_1 = \frac{(120d)(5\Delta t)}{2h}.$$

Taking the modulus of Eq.(9) gives $|g| < 1$; therefore the linearised scheme is unconditionally stable.

THE INITIAL STATE

From the initial condition U(x,0) on the function U(x,t) we must determine the initial vector $\mathbf{d}^0$ so that the time evolution of $\mathbf{d}$, using (7), can be started. Firstly rewrite Eq.(2) for the initial condition as

$$U_N(x,0) = \sum_{j=-2}^{N+2} \delta_j^0 \phi_j(x) \tag{10}$$

where δ_j^0 are unknown parameters to be determined. To do this we require $U_N(x,0)$ to satisfy the following constraints:

(a) It must agree with the initial condition U(x,0) at the knots x_j, $j=0,1,...,N$.

(b) The first and the second derivatives of the approximate initial condition agree with those of the exact initial condition at both ends of the range; Eq.(3) produces two further equations.

The initial vector $\mathbf{d}^0$ is then determined as the solution of a matrix equation derived from Eq.(3)

$$\mathbf{M}\,\mathbf{d}^0 = \mathbf{b} \tag{11}$$

THE TEST PROBLEMS

We will now validate our algorithm by studying the motion of solitary waves . It is well known that Eq.(1) has a two parameter analytic solution of the form (Gardner)

$$U(x,t) = b + 3c\ \mathrm{sech}^2(k[x-x_0-(b+c)t]) \tag{12}$$

where $k = \frac{1}{2}\sqrt{\frac{c}{v(b+c)}}$ and b and c are constants.

This solution with b = 1 is physically valid and corresponds to that used by Eilbeck and McGuire and Samtarelli and applies to a single solitary wave of magnitude 3c, initially centered on x_0, propagating to the right without change of shape at a steady velocity (1+c).

Olver has shown that the **RLW** equation possesses only three polynomial invariants. We will examine the conservation properties of the algorithm by calculating these invariants, which for the **RLW** equation in the form (1) are:

$$C_1 = \int_a^b U dx,\ C_2 = \int_a^b (U^2 + vU_x^2) dx \text{ and } C_3 = \int_a^b U^3 dx.$$

We now discuss the following cases:

(a) We consider the motion of a single solitary wave and take the initial condition

$$U(x,0) = 1+3c\ \text{sech}^2(Ax+D)$$

with $c = 0.3$, $v = 1.0$, $A = \frac{1}{2}\sqrt{\frac{c}{v(1+c)}}$, and $D = -40A$. The range $0 \leq x \leq 80$ is divided into 400 elements of equal length h=0.2 and a time step Δt=0.1 used. We observe the solitary wave moving to the right unchanged in form and with a velocity c = 1.3.

Table (2) : Single soliton $h = 0.2$, $\Delta t = 0.1$, $0 \leq x \leq 80$, $v = 1.0$

Time	Galerkin with cubic spline(Gardner)	collocation Quintic
	L_2-Norm$\times 10^3$	L_2-Norm$\times 10^3$
1.0	0.199	0.174
1.5	0.289	0.260
2.0	0.378	0.346
2.5	0.472	0.430
3.0	0.565	0.514
3.5	0.657	0.596
4.0	0.747	0.678
4.5	0.832	0.758
5.00	0.901	0.836

From Table 2 we notice that L_2 norm calculated by our scheme is more accurate than that obtained by Gardner

Table (3) : Invariants for single soliton

Time	Gardner	Our scheme	Gardner	Our scheme	Gardner	Our scheme
	C_1		C_2		C_3	
0.5	87.4941	87.6940	99.6922	99.8919	119.2089	119.4085
1.0	87.4942	87.6940	99.6923	99.8919	119.2091	119.4085
1.5	87.4944	87.6940	99.6927	99.8919	119.2095	119.4086
2.0	87.4945	87.6940	99.6930	99.8919	119.2100	119.4086
2.5	87.4947	87.6940	99.6943	99.8920	119.2104	119.4086
3.0	87.4949	87.6940	99.6936	99.8919	119.2110	119.4086
3.5	87.4951	87.6940	99.6940	99.8920	119.2114	119.4087
4.0	87.4953	87.6940	99.6943	99.8920	119.2118	119.4087
4.5	87.4955	87.6940	99.6948	99.8920	119.2125	119.4087
5.0	87.4957	87.6940	99.6951	99.8920	119.2130	119.4087

Table 3 shows us that in our scheme the change in the values of the quantities C_1, C_2 and C_3 during the computer run are satisfactorily constant, each changes less than 2×10^{-4}, but in the Galerkin method (Gardner) the changes in these quantities are less than 5×10^{-3} at $h = 0.2$ and $\Delta t = 0.1$.

(b) We have examined the evolution of an initial Maxwellian pulse into solitary waves, using as initial condition

$$U(x,0) = 1 + \exp(-(x-7)^2)$$

For $\nu = 0.04$ the Maxwellian develops into a single solitary wave with magnitude and velocity consistent with equation (12), plus a well developed oscillating tail. This results bears a strong resemblance to the corresponding **KDV** simulation. The values of the quantities C_1, C_2 and C_3 are given in Table 4

From Table 4 we find that the changes in C_1, C_2 and C_3 were by a factor of 0.9×10^{-3}, 1.0×10^{-2} and 0.2×10^{-3} respectively in the Grlerkin method [10], but the changes in our scheme by a factor less than 1.0×10^{-4}, 1.0×10^{-4} , 5.0×10^{-4} respectively.

Table (4) : $\nu = 0.04$

Time	Gardner C_1	Our scheme C_1	Gardner C_2	Our scheme C_2	Gardner C_3	Our scheme C_3
2.0	31.7730	31.9724	34.8492	35.0483	40.1019	40.3024
4.0	31.7733	31.9724	34.8501	35.0483	40.1029	40.3029
6.0	31.7737	31.9724	34.8507	35.0483	40.1037	40.3029
8.0	31.7739	31.9724	34.8509	35.0482	40.1039	40.3029

When $\nu = 0.01$ the final state is composed of 2 solitary waves each of which has magnitude and velocity consistent with equation (12) breakup into solitary waves is not clean however as a small disturbance.

Table (5) : $\nu = 0.01$

Time	Gardner C_1	Our scheme C_1	Gardner C_2	Our scheme C_2	Gardner C_3	Our scheme C_3
2.0	31.7725	31.9724	34.8108	35.0108	40.1010	40.3177
4.0	31.7724	31.9724	34.8106	35.0107	40.1011	40.3377
6.0	31.7721	31.9724	34.8102	35.0107	40.1006	40.3421
8.0	31.7719	31.9724	34.8097	35.0106	40.1000	40.3425
10.	31.7718	31.9724	34.8095	35.0105	40.0996	40.3426
12.	31.7718	31.9723	34.8094	35.0104	40.0944	40.3425
14.	31.7718	31.9723	34.8094	35.0103	40.0995	40.3423

From Table 5 we show that the invariants C_1, C_2 and C_3 are changes by 1.0×10^{-3}, 1.0×10^{-2}, and 1.0×10^{-1} respectively in Galerkin method (Gardner), but these quantities have been changed during the computer run by 1.0×10^{-4}, 5.0×10^{-4}, and 2.0×10^{-2} respectively in the present method.

From Table 6 the invariants C_1, C_2, and C_3 are changes by $1.0\text{x}10^{-4}$, $8.0\text{x}10^{-3}$, and $1.0\text{x}10^{-1}$ respectively in Galerkin method (Gardner), but these quantities have been changed during the computer run by $2.0\text{x}10^{-4}$, $5.0\text{x}10^{-4}$, and $3.0\text{x}10^{-2}$ respectively in the our scheme.

Table (6): ν = 0.001

Time	Gardner C_1	Our scheme C_1	Gardner C_2	Our scheme C_2	Gardner C_3	Our scheme C_3
2.0	31.7725	31.9725	34.8016	35.0108	40.1010	40.3177
4.0	31.7724	31.9725	34.8016	35.0107	40.1011	40.3377
6.0	31.7724	31.9724	34.8017	35.0107	40.1006	40.3421
8.0	31.7724	31.9724	34.8097	35.0106	40.1000	40.3425
10.	31.7724	31.9724	34.8095	35.0105	40.0996	40.3426
12.	31.7725	31.9723	34.8094	35.0104	40.0944	40.3425
14.	31.7725	31.9723	34.8094	35.0103	40.0995	40.3423

CONCLUSION

We have shown that the finite element method used in this paper can faithfully represent the amplitude, position and velocity of a single solitary wave. The L_2-Norm calculated by our scheme is very small compared with that calculated by the Galerkin method [10]. The three that invariants of motion are satisfactorily constants in all the computer simulations described here, so that the algorithm can fairly describe the invariant quantities as conservative. The numerical scheme has been shown to be unconditionally stable. We have further shown that the algorithm copes well with the generation of solitary waves from an arbitrary initial pulse, and conclude that it may widely be used for runs of the **RLW** equation for long duration.

We have demonstrated that using quintic splines is easy to apply as element shape and weight functions. We believe that this approach will be useful also for other applications where the continuity of derivatives is essential.

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طريقة حسابية لحل معادلة الموجة الطويلة المنتظمة

أحمد حسن أحمد على

هذا البحث يعتمد أساساً على طريقة العنصر المحدود لحل معادلة الموجة الطويلة المنتظمة وهى معادلة تفاضلية جزئية غير خطية وذلك لإهميتها في العديد من التطبيقات الفيزيائية. وقد إسخدمنا إحدى طرق العنصر المحدود وهى طريقة نقاط الملاءمة التى تعتمد على الوصلات الخماسية الشكل. وقد تم عمل مقارنة بين النتائج التى توصلنا إليها والنتائج السابقة ووجد أن الطريقة التى إستخدمناها أكثر دقة وأكثر كفاءة. ولهذا نوصى بتطبيق هذه الطريقة على المعادلة المشابه لها.