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ORTHOGONAL DOUBLE COVERS OF COMPLETE BIPARTITE GRAPH BY THE UNION OF CERTAIN STARS

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ABSTRACT

An Orthogonal Double Cover (ODC) of the complete graph is a set of graphs such that every two of them share exactly one edge and every edge of the complete graph belongs to exactly two of the graphs. We consider the case where the graph to be covered twice is the complete bipartite graphs, and all graphs in the collection are isomorphic to the spanning subgraph G (union of certain stars).

الغطاء العمودي المزدوج (ODC) للأشكال الكاملة (K_n) عبارة عن فئة من الأشكال الجزئية حيث:

- كل شكلين منهم يشتركان في حافة واحدة.
- وكل حافة من حواف الشكل الكامل تنتمي إلي شكلين فقط من فئة الأشكال الجزئية.

في هذا البحث تم إنشاء الأشكال المولدة للأغطية المتعامدة المزدوجة للشكل الثنائي الكامل $(K_{n,n})$ باستخدام أشكال مشابه للشكل G (عبارة عن اتحاد مجموعة من أشكال النجوم).

Keywords: Orthogonal double cover; ODC; Graph decompositions

1. INTRODUCTION

Let G be a collection of n spanning subgraphs (called pages) of K_n (the complete graph on n vertices). G is called an *ODC* if

1. Every edge of K_n is an edge in exactly two of the pages,
2. Any two pages share exactly one edge.

If all pages in G are isomorphic to a graph *G* then G is said to be an ODC by *G*. Clearly, *G* must have exactly *n-1* edges. The existence of an ODC was considered by many authors [1-10]. Although progress on the existence problem has been made it is far from being solved completely [9].

A generalization of the notion of an *ODC* to arbitrary underlying graphs is as follows. Let H be an arbitrary graph with n vertices and let $G=\{G_0,\ldots,G_{n-1}\}$ be a collection of *n* spanning subgraphs of H. G is called an *ODC* of H if there exists a bijective mapping $\varphi: V(H) \to G$ such that:

(i) Every edge of H is contained in exactly two of the graphs $G_0,\ldots,G_{n-1}$.

(ii) For every choice of different vertices a,b of H,

$$|E(\varphi(a)) \cap E(\varphi(b))| = \begin{cases} 1 & if\ \{a,b\} \in E(H), \\ or & \\ 0 & otherwise. \end{cases}$$

We consider the complete bipartite graph $K_{n,n}$ on $2n$ vertices. The vertices of $K_{n,n}$ are labeled by the elements of $\Gamma \times Z_2$, where $\Gamma = \{\gamma_0,\ldots,\gamma_{n-1}\}$ is an additive group of order *n*, such that $\{w_i,u_i\} \notin E(K_{n,n})$ for $w,u \in \Gamma$ and fixed $i \in Z_2$. Here, The groups Γ and Z_2 are used just for labeling the vertices.

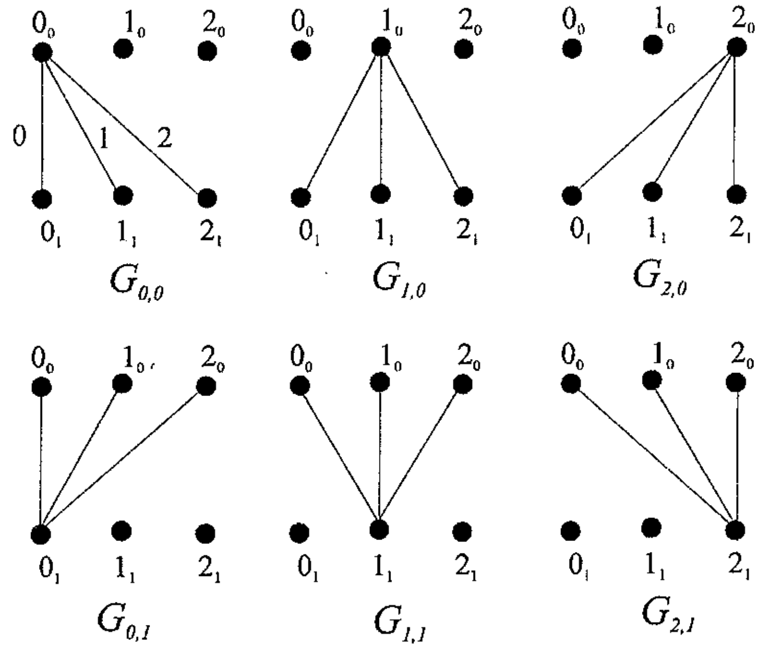

Figure 1: *ODC* of $K_{3,3}$ *by* S_3, $\Gamma = Z_3$.

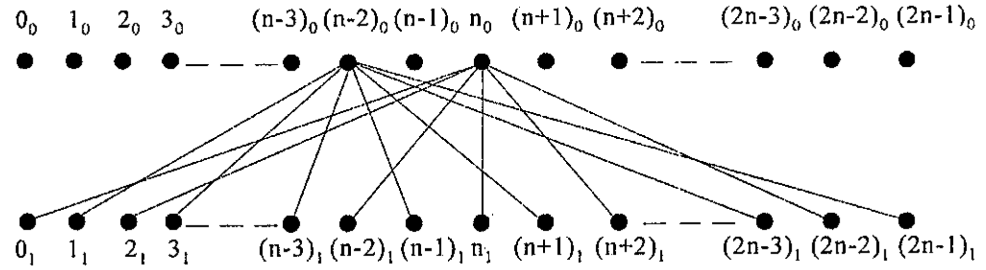

Figure 2: Symmetric half starter of an *ODC* of $K_{2n,2n}$ *by* $2S_n$, *with* $\Gamma = Z_{2n}$, n is *even.*

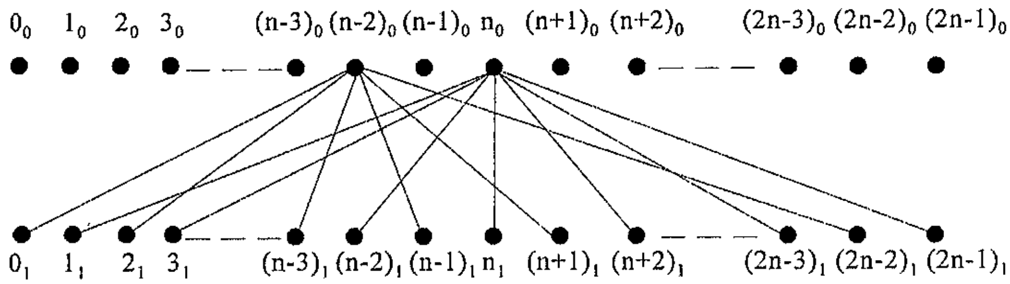

Figure 3: Symmetric half starter of an *ODC* of $K_{2n,2n}$ *by* $2S_n$, *with* $\Gamma = Z_{2n}$, n is *odd.*

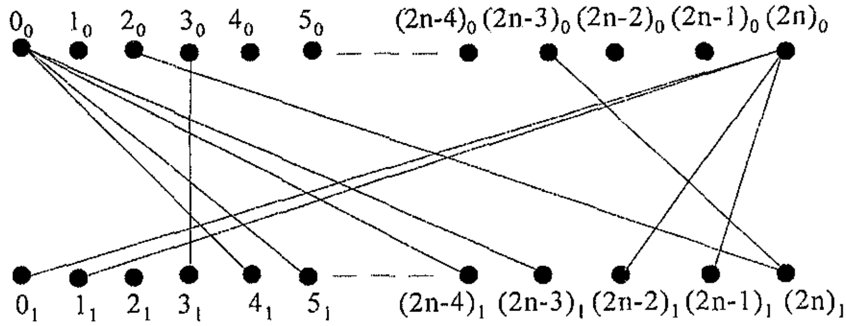

Figure 4: Symmetric half starter of an *ODC* of $K_{2n+1,2n+1}$ by $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6}$, *with* $\Gamma = Z_{2n+1}$.

Abelian groups, which are represented by Cartesian product of cyclic groups, are considered. Therefore, the lexicographic order is used, i.e. in case of $\Gamma = Z_n$ it is just the natural order $0, 1, \ldots, n-1$. From the double cover property *(i)* it follows that G contains exactly n edges, and the orthogonality property (ii) forces the graphs $G_{w,0}$, *and*, $G_{w,1}$ for all $w \in \Gamma$ to form two orthogonal edge decompositions of $K_{n,n}$.

In Figure 1 an *ODC* of $K_{3,3}$ by $G = S_3$ is exhibited. In this paper we have constructed the *ODC* of $K_{2n,2n}$ by $G = 2S_n$ (vertex disjoint union of two stars), and $K_{2n+1,2n+1}$ by $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6}$ (vertex disjoint union of stars) which will be considered in the following section.

2. ODC OF $K_{n,n}$ BY SYMMETRIC STARTERS

Let G be a spanning subgraph of $K_{n,n}$ and $a \in \Gamma$. Then the graph G with $E(G+a) = \{(u+a, v+a) : (u,v) \in E(G)\}$ is called the *a-translate* of G. The length of an edge $e = (u,v) \in E(G)$ is defined by $d(e) = v - u$ (as shown in Figure 1 the edges of $G_{0,0}$ are labeled by its length).

G is called a half starter with respect to Γ *if* $|E(G)| = n$ and the lengths of all edges in G are mutually different, i.e. $\{d(e) : e \in E(G)\} = \Gamma$.

Theorem 1 (see [7]). *If* G *is a half starter, then the union of all translates of* G *forms an edge decomposition of* $K_{n,n}$.

$$\bigcup_{a \in \Gamma} E(G+a) = E(K_{n,n}).$$

Here, the half starter will be represented by the vector: -

$$v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}}).$$

Where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i. Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 2 (see [7]). *If two half starters* $v(G_0)$ *and* $v(G_1)$ *are orthogonal, then* $G = \{G_{a,i} : (a,i) \in \Gamma \times Z_2\}$ *with* $G_{a,i} = (G_i + a)$ *is an ODC of* $K_{n,n}$.

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{u_0, v_1\} : \{v_0, u_1\} \in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a symmetric starter with respect Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 3 (see [7]). *Let* n *be a positive integer and let* G *be a half starter represented by*

$v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$. *Then* G *is symmetric starter if* $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.

For all positive integers $n \geq 2$, let us define the graph $G = 2S_n$ (vertex disjoint union of two stars) to be a spanning subgraph of the complete bipartite graph $K_{2n,2n}$, where there are two cases:

Firstly, when n is even, as shown in Figure 2, we have

$$E(2S_n) = \{((n-2)_0, j_1) : j \text{ is odd}\} \cup \{((n)_0, j_1) : j \text{ is even}\}.$$

Secondly, when n is odd, as shown in Figure 3, we have

$$E(2S_n) = \{((n-2)_0, j_1) : j \text{ is even}\} \cup \{((n)_0, j_1) : j \text{ is odd}\}.$$

Then we can construct the following theorem:-

Theorem 4. *For all positive integers* $n \geq 2$, *let* $\Gamma = Z_{2n}$. *Then the vector* $v(G) = (n, n-2, n, n-2, n, n-2, \ldots, n, n-2)$ *is a symmetric starter of an ODC of* $K_{2n,2n}$ *by* S_{2n}.

Proof. From the vector $v(G) = (n, n-2, n, n-2, n, n-2, \ldots, n, n-2)$ we can define:

$$v_i(G) = v_{-i}(G) = \begin{cases} n & \text{if } i \text{ is even, or} \\ n-2 & \text{if } i \text{ is odd.} \end{cases}$$

Then for all $i \in Z_{2n}$, we have

$$v_i(G) - v_{-i}(G) + i = i.$$

Applying Theorem 3 proves the claim. Thus, for any $i \in \Gamma = Z_{2n}$ the i^{th} graph isomorphic to the the graph $G = 2S_n$ has the edges:

$$E(G_i) = \begin{cases} \{((i+n-2)_0, (i+j)_1) : j \text{ is odd}\} \\ \cup\{((i+n)_0, (i+j)_1) : j \text{ is even}\} \\ \qquad \text{if } n \text{ is even, or} \\ \{((i+n-2)_0, (i+j)_1) : j \text{ is even}\} \\ \cup\{((i+n)_0, (i+j)_1) : j \text{ is odd}\} \\ \qquad \text{if } n \text{ is odd.}\Box \end{cases}$$

For any positive integer $n \geq 4$, $\Gamma = Z_{2n+1}$. Let us define the graph $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6}$ (vertex disjoint union of stars) to be a spanning subgraph of $K_{2n+1,2n+1}$ as shown in Figure 4. Then we can construct the following theorem: -

Theorem 5. *For any positive integer* $n \geq 4$, *let* $\Gamma = Z_{2n+1}$ *then the vector* $v(G) = (3, 2n, 2n, 2n-3, 0, 0, \ldots, 0, 0, 2, 2n, 2n)$ *is a symmetric starter of an ODC of* $K_{2n+1,2n+1}$ *by* $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6}$.

Proof. For any positive integer $n \geq 4$, let $\Gamma = Z_{2n+1}$, then from the vector

$v(G) = (3, 2n, 2n, 2n-3, 0, 0, \ldots, 0, 0, 2, 2n, 2n)$ we can define

$$v_i(G) = v_{-i}(G) = \begin{cases} 3 & \text{if } i = 0, or \\ 2n & \text{if } i = 1, 2, \\ & \quad 2n-1, 2n, or \\ 2n-3 & \text{if } i = 3, or \\ 2 & \text{if } i = 2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $i \in \Gamma = Z_{2n+1}$ we have

$$v_i(G) + v_{-i}(G) + i = \begin{cases} -i & \text{if } i = 3, 2n-2, or \\ i & \text{otherwise.} \end{cases}$$

Applying Theorem 3 proves the claim. Note that for any $i \in \Gamma = Z_{2n+1}$, the i^{th} graph isomorphic to $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6}$ has the edges:

$$\{((i+2n)_0, (i+j)_1) : j = 0, 1, 2n-2, 2n-1\} \cup \{((i+j)_0, (i+2n)_1) : j = 2, 2n-3\} \cup \{((i+3)_0, (i+3)_1)\} \cup \{(i_0, (i+j)_1) : 4 \leq j \leq 2n-3\}\Box$$

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