Local Similarity Solution Of The Free Convection From A Long Vertical Cylinder Embedded In A Darcian Fluid-Saturated Porous Medium

الحل التشابهي المحلي للحمل الحر من أسطوانة رأسية طويلة مطمورة في وسط مسامي مشبع

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خلاصة: في هذا البحث أمكن الحصول على حل محلي تشابهي للحمل الحر من أسطوانة رأسية طويلة مطمورة في وسط مسامي مشبع. أفترض أن سطح الاسطوانة محفوظ عند درجة حرارة ثابتة. تم وصف الحمل الحر بواسطة كل من معادلة السريان و معادلتي دارسي للحركة و معادلة الطاقة باستخدام نظام الإحداثيات الأسطوانية. للحصول على الحل التشابهي تم وضع المعادلات الواصفة في صورتها اللابعدية وذلك بتعريف مناسب لكل من المتغيرات المستقلة والتابعة. لما كانت المعادلات الواصفة في صورتها اللابعدية عبارة عن مجموعة من المعادلات التفاضلية العادية فقد تم حلها عدديا بواسطة طريقة رونج كتا مصحوبة بطريقة الرصد لحل مسائل القيمة المحددة. بحل هذه المعادلات عند مواضع مختلفة بطول الأسطوانة أمكن الحصول على كل من توزيع درجات الحرارة و السرعة وكذلك رقم نوسلت المحلي بطول الأسطوانة. حيث أن نموذج دارسي قد استخدم في وصف الحركة فإن هذا البحث يكون مناسب لحالات الحمل الحر في الأوساط ذات مسامية صغيرة. تم حساب رقم نوسلت المحلي بطول الأسطوانة لقيم رقم رايلي ١، ٥، ١٠، ٢٠.

Abstract In this work, local similarity solution is obtained for free convection from a long cylinder embedded in a fluid-saturated porous medium. The cylinder wall is maintained constant. The free convection along the cylinder is described by continuity equation, momentum equation in axial direction, momentum equation in radial direction and energy equation. These governing equations are expressed in cylindrical coordinates. Defining new proper independent and dependent variables, the governing equations are transformed to dimensionless form. Since the proposed solution is restricted to the case of long cylinders, the derived dimensionless governing equations are ordinary differential equations of the boundary value problems type. They are solved numerically using Runge-Kutta method accompanied with shooting technique. Newton-Raphson method of non-algebraic equation is used to carry out the shooting technique. Solving these equations at different positions along the cylinder, temperature and velocity distributions are obtained and hence the values of local Nusselt number are calculated for Rayleigh number of 1, 5, 10 and 20. Because of the application of Darcian model, in this work, the obtained results are valid only for the case of porous medium of small permeability.

1. Introduction

Natural convection heat transfer in a fluid-saturated porous media is of great interest because of its numerous practical applications. Thermal insulation, chemical reactors, underground spread of pollutants and geophysical problems are examples of these applications. Hsieh et al. [1] reported a nonsimilarity solutions for mixed convection from a vertical flat plate embedded in a porous medium. Both surface heating-conditions of variable wall temperature and of variable heat flux were studied. Correlations for local and average Nusselt numbers were presented. Non-Darcy mixed convection along nonisothermal vertical surfaces in porous media was studied by Chien-Hsin et al. [2]. In this work, entire mixed convection regime is covered by a single parameter. A finite difference scheme was used to solve the transformed system of equations. Mixed convection from vertical cylinder embedded in a porous medium was studied by Aldoss et al. [3]. Nonsimilarity solutions are obtained for the case of variable wall temperature and variable surface heat flux. The effect of characteristic parameters of the problem on heat transfer is investigated.

Natural convection in a porous medium is a point of interest for many investigators. Non-darcy natural convection around a horizontal cylinder buried near the surface of a fluid-saturated porous medium was studied by Christopher et al. [4]. The governing equations are solved numerically to obtain the flow field and the temperature distribution around the cylinder. Local and average Nusselt numbers are expressed as functions of cylinder depth, the

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modified Rayleigh number and Darcy number. Leu and Jhn-Yuhjang [5] studied natural convection from a point heat source embedded in a non-Darcian porous medium. Local similarity and modified Keller's Box methods are employed. Natural convection heat transfer between two porous media separated by a vertical wall was studied by Higuera and Pop [6]. In this work, the problem of coupled heat transfer by natural convection between two fluid-saturated porous media at different temperatures separated by a vertical conductive wall is investigated analytically and numerically, taking in account the two-dimensional thermal conduction in the separating wall. Higuera [7] studied the conjugate heat transfer across a thin horizontal wall separating two fluid-saturated porous media at different temperature. Natural convection heat transfer from an isothermal vertical surface to a fluid-saturated thermally stratified porous medium was studied by Angirasa and Peterson [8]. They presented the results of a numerical study of natural convection heat transfer in a stable stratified, fluid-saturated low porosity medium. In this investigation, the boundary layer approximations are described and a wide range of ambient thermal stratification levels are considered.

In the present work, attempt is made to obtain a local similarity solution for free convection from a long constant wall-temperature cylinder surrounded by a Darcian fluid-saturated porous medium. According to the presented solution, the problem parameters are reduced to single characteristic parameter; the Rayleigh number based on the cylinder radius.

2. Description of Mathematical Model

The description of the problem and the coordinate system used to investigate the free convection induced due to hot embedded cylinder in a fluid-saturated porous medium is shown in figure (1). The problem is described by the differential form of conservation laws of mass, momentum, and energy in cylindrical coordinates. It is proper to consider the flow to be axisymmetric and accordingly, the tangential component of the velocity and its derivatives vanish. Moreover the derivatives with respect to angular displacement also vanish. According to the foregoing assumptions, the governing equations of convective flow can be written as the following;

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\, v_r\right)+\frac{\partial v_z}{\partial z}=0 \qquad , \qquad (1)$$

$$v_r=-\frac{K}{\mu}\frac{\partial p}{\partial r} \qquad , \qquad (2)$$

$$v_z=-\frac{K}{\mu}\left(\frac{\partial p}{\partial z}-\rho g\right) \qquad , \qquad (3)$$

$$v_r\frac{\partial T}{\partial r}+v_z\frac{\partial T}{\partial z}=\alpha\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad , \qquad (4)$$

where K and μ are the permeability of porous medium and dynamic viscosity; respectively. p, ρ, α and g are the pressure, density, thermal diffusivity and gravitational acceleration; respectively. Volumetric-averaged radial and axial velocity components and temperature are denoted as v_r , v_z and T; respectively. The physical properties of the medium are assumed to be isotropic and both fluid and solid matrix of the medium are assumed to be in thermal equilibrium [9]. In energy equation (4), the conductive heat in axial direction is neglected

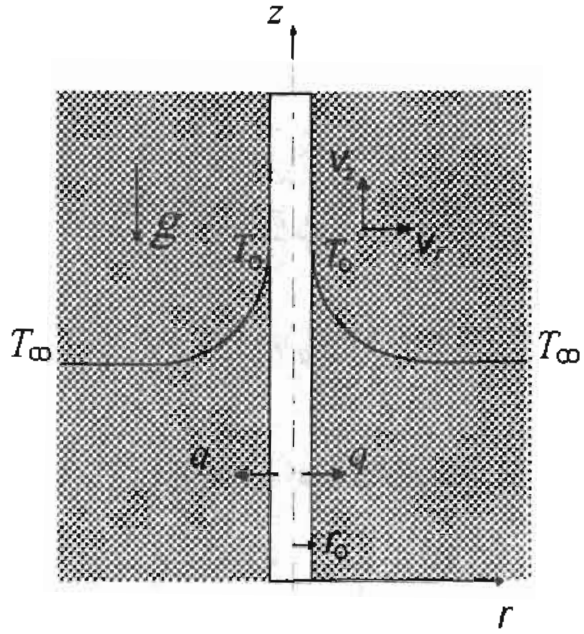

Figure (1) Physical description of the problem

compared with that in radial direction. This assumption is valid in case of long cylinder (length of the cylinder >> its radius). To eliminate the pressure, equations of motion (2 and 3) are differentiated with respect to z and r; respectively and with some manipulations, they can be reduced to a single equation;

$$\frac{\partial v_z}{\partial r} - \frac{\partial v_r}{\partial z} = \frac{K g}{\mu} \frac{\partial \rho}{\partial r} \tag{5}$$

One can use the definition of the coefficient of thermal expansion β

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taking in account Boussineq approximation to modify the equation of motion (5) to;

$$\frac{\partial v_z}{\partial r} - \frac{\partial v_r}{\partial z} = - \frac{K g \beta}{\nu} \frac{\partial T}{\partial r} \tag{6}$$

Equations (1, 4 & 6) are the governing equations of the problem, they must satisfy the following boundary conditions;

at $r = r_o$ $\quad v_r = v_z = 0$ $\quad$ and $\quad T = T_o = const.$

at $r \to \infty$ $\quad v_r = 0$ $\quad$ and $\quad T = T_\infty = ambient\ temp.$

Moreover, one can define the stream function ψ such that it satisfies the continuity equation (1). Accordingly ψ is defined as;

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{7}$$

Substitution with (7) in equations (4 & 6) leads to;

$$\frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \psi}{\partial r} = -\frac{K g \beta}{\nu}\frac{\partial T}{\partial r} \quad , \tag{8}$$

$$-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial T}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right) \tag{9}$$

Equations (8-9) must satisfy the following boundary conditions;

$$\text{at } r = r_o \qquad \frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial z} = 0 \; ; \qquad T = T_o \quad ,$$

$$\text{at } r \to \infty \qquad \frac{\partial \psi}{\partial z} = 0 \qquad ; \qquad T = T_\infty \quad . \tag{10}$$

In modified momentum equation (8) the second derivative with respect to z is neglected. This carried out simplification is seams to be reasonable because the height of the examined cylinder is long enough such that this derivative is very small compared with the derivative with respect to r ($\frac{\partial^2 \psi}{\partial z^2} << \frac{\partial^2 \psi}{\partial r^2}$). Solving equations (8-10), the physical quantities of interest can be evaluated such as velocity, temperature distribution and Nusselt number. Local Nusselt number is defined according to the following relations;

$$Nu_z = \frac{h\, z}{k} = \frac{q_o\, z}{k\,(T_o - T_\infty)} = \frac{-\left.\frac{\partial T}{\partial r}\right)_{wall} z}{(T_o - T_\infty)} \quad , \tag{11}$$

where Nu_z and q_o are local Nusselt number and heat flux at the cylinder wall. In order to put the governing equations (8-10) in dimensionless form, the following dependent and independent variables are proposed;

$$\eta = r\sqrt{\frac{K g \beta\,(T_o - T_\infty)}{\alpha \nu z}} \; ; \; f(\eta) = \frac{\psi(r,z)}{\alpha z} \; ; \; \theta(\eta) = \frac{T(r,z) - T_\infty}{T_o - T_\infty} \quad , \tag{12}$$

where η , f and θ are the local similarity variable, dimensionless stream function and temperature. Accordingly the dimensionless form of governing equations (8-10) can , with the aid of relations (12), be derived as;

$$\eta\, f_{\eta\eta} + f_\eta + \eta^2\, \theta_\eta = 0 \qquad , \tag{13}$$

$$\eta\, \theta_{\eta\eta} + (1+f)\, \theta_\eta = 0 \qquad , \tag{14}$$

where the prefix η denotes the differentiation with respect to η. Equations (13-14) must satisfy the following modified boundary conditions;

$$\begin{aligned} &\text{at } \eta = \eta_o \;; \quad f = f_\eta = 0 \quad \text{and} \quad \theta = 1 \quad , \\ &\text{at } \eta \to \infty \;; \quad \theta = 0 \quad , \end{aligned} \tag{15}$$

where η_o is the value of the independent variable η at the cylinder wall, which is defined as follows;

$$\eta_o = r_o \sqrt{\frac{K\, g\, \beta\, (T_o - T_\infty)}{\alpha\, \nu\, z}} = \sqrt{\frac{r_o}{z}} \cdot \sqrt{Ra_o} \qquad , \tag{16}$$

where Ra_o is Rayleigh number based on the cylinder radius r_o and is defined as;

$$Ra_o = \frac{K\, g\, \beta\, (T_o - T_\infty)\, r_o}{\alpha\, \nu} \tag{17}$$

Referring to equations (13-15), the problem variables (f and θ) are function of single independent variable (η) whatever the value of z and r is. Besides η, the value of it at the cylinder surface (η_o) is required to carry out the solution of the governing equations. Accordingly, the obtained solution is local similarity solution. Equations (13 and 14) are ordinary differential equations of boundary-value problems type. These equations are solved, numerically, using the well known Runge-Kutta method for ordinary differential equations accompanied with shooting method of boundary value problems. According to this technique, equations (13 and 14) are transformed to a set of six first order ordinary differential equations. These equations are solved, simultaneously, at different values of η_o (at different positions along the cylinder). Solving the mentioned equations, one can obtain the dimensionless stream function (f) and temperature (θ) and their derivatives as functions of η throughout the flow field. Using equations (7,11and 12) one can express dimensionless radial and axial components of velocity (V_r & V_z) and local Nusselt number Nu_z as functions of η, f, θ according to the following relations;

$$V_r = v_r \,/ \left(\frac{\alpha}{z}\, \sqrt{Ra_z}\,\right) = \frac{1}{2}\, f_\eta - \frac{1}{\eta}\, f \qquad ,$$

$$V_z = v_z \Big/ \left(\frac{\alpha}{r} \sqrt{Ra_z} \right) = f_\eta \quad , \qquad (18)$$

$$Nu_z \Big/ \sqrt{Ra_z} = -\theta_\eta \,)_{wall}$$

Where $\theta_\eta\,)_{wall}$ is the derivative of θ with respect to η at the cylinder wall. Ra_z is the local Rayleigh number based on z and is defined as;

$$Ra_z = \frac{K\ g\ \beta\ (T_o - T_\infty)\ z}{\alpha\ \nu}$$

3. Results and discussion

The dimensionless ordinary differential equations (13 and 14) were solved by Runge-Kutta method accompanied with shooting method using Newton-Raphson method of non-algebraic equations. The suitable step size of η was found to be 0.025 for all carried out runs and the proper maximum value of η corresponding to $r \to \infty$ was found to be 10. The solution was carried out, separately, for different numerical values of η_o. Figures (2 and 3)

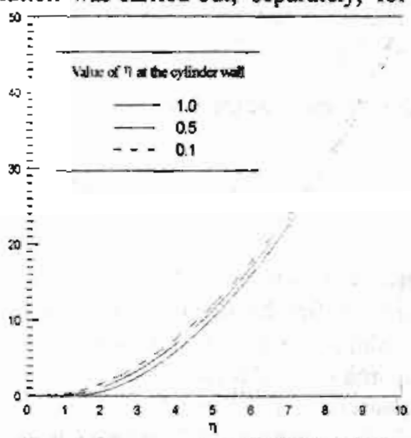

Figure (2) Dimensionless stream function versus local similarity variable η

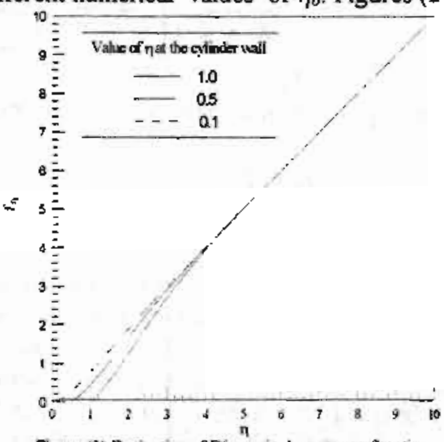

Figure (3) Derivative of Dimensionless stream function versus the local similarity variable

show the dimensionless stream function (f) and its derivative (f_η) versus η at different values of η_o (at different positions along the cylinder). According to the definition of η_o equation (16) and for specified cylinder (r_o = constant), the distance measured along the cylinder z decreases as the value of η_o increases. As it is clear in figure (2), the value of f is higher with decreasing η_o (and in turn; increasing z). From figure (3), f_η increases as η_o decreases (z increases) until $\eta \sim 5.0$. In general, f and f_η increase with increasing η. Figure (4) shows the dimensionless axial velocity as it is defined through equation (18) against η. This velocity takes an asymptotic value of 1.0 whatever the value of η_o is. Dimensionless radial component of velocity as it is defined through equation (18); is shown in figure (5). This velocity has a peak-value, which increases for higher values of η_o. Both the dimensionless temperature θ and its derivative θ_η are shown in figures (6 and 7). They go to an asymptotic value of zero for all values of η_o.

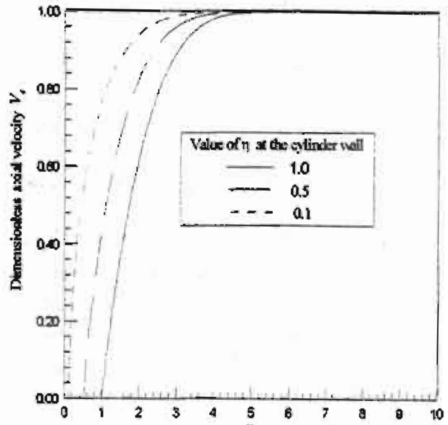

Figure (4) Dimensionless axial component of velocity against the local similarity variable η

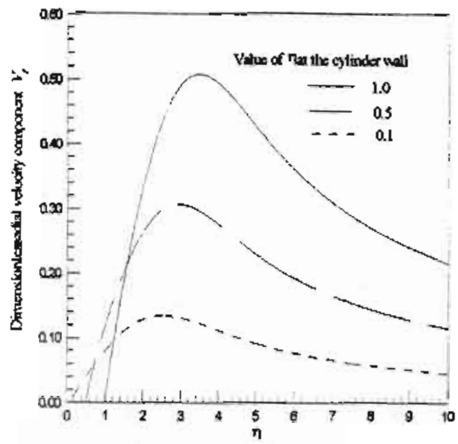

Figure (5) Dimensionless radial velocity component versus local similarity variable η

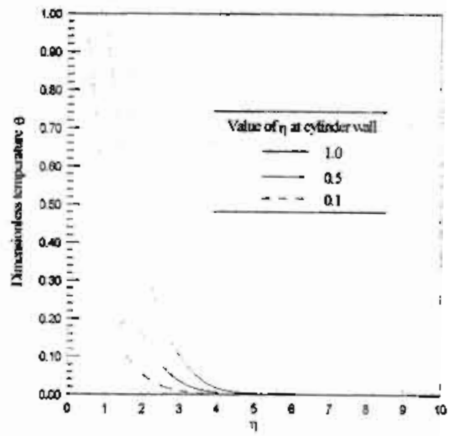

Figure (6) Dimensionless temperature versus the local similarity variable η

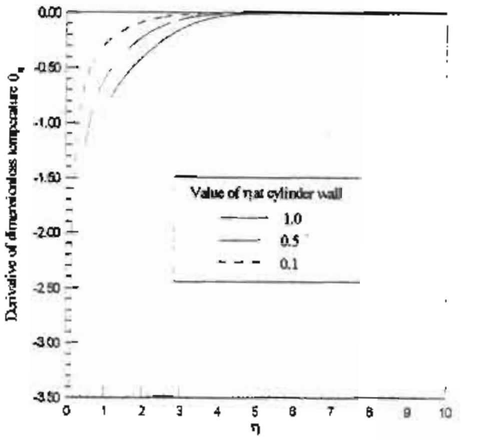

Figure (7) Derivative of dimensionless temperature versus local similarity variable η

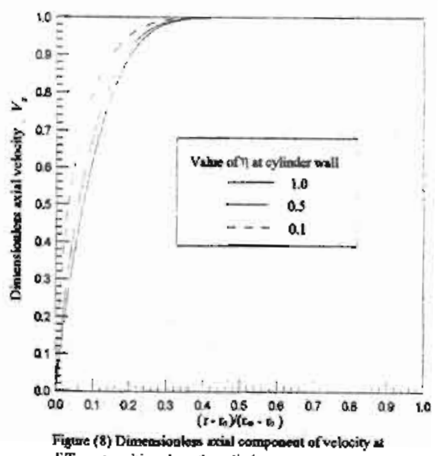

Figure (8) Dimensionless axial component of velocity at different position along the cylinder

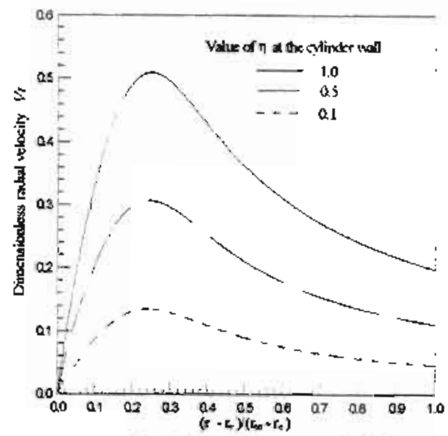

Figure (9) Dimensionless radial component of velocity at different axial positions

Figures (8-10) show the dimensionless axial velocity, radial velocity and temperature versus dimensionless radial position ($\frac{r-r_o}{r_\infty - r_o} = \frac{\eta - \eta_o}{\eta_{max} - \eta_o}$) for different values of η_o. Dimensionless axial velocity [see equation (18)] increases rapidly then gradually till it reaches an asymptotic value of 1.0 (starting from $\frac{r-r_o}{r_\infty - r_o} \approx 0.4$).For all values of η_o, dimensionless radial velocity has a peak at the same dimensionless radial position of about 0.24 for all values of η_o. From figure (10); the dimensionless temperature decreases rapidly till radial position of 0.2, there it decreases slowly till it reaches an asymptotic value of zero at dimensionless radial position of about 0.4. In accordance, the thermal flow field terminates at this position, thereafter the gravitational force is, solely, active. As it is expected, the flow starts to decelerate till its axial and radial components of velocity vanish at the end of the hydrodynamic flow field. This portion of hydrodynamic flow field is not considered in the present work (see figures (4,5,8 and 9).

Figure (11) shows local Nusselt number ($Nu_z / \sqrt{Ra_z}$) along the cylinder length (at different values of η_o). As η_o decreases Nusselt number increases and goes to infinity as $\eta_o \to 0$. It decreases rapidly for smaller values of η_o [$\eta_o = O(0.0)$], then it decreases gradually. Using equations (16-18), one can derive dimensionless distance along the cylinder (z/r_o) and local Nusselt number based on r_o (Nu_o) as;

$$\frac{z}{r_o} = \frac{1}{(\eta_o)^2} \cdot Ra_o \;; \qquad Nu_o = \frac{h\, r_o}{k} = \frac{r_o}{z} \cdot Nu_z$$

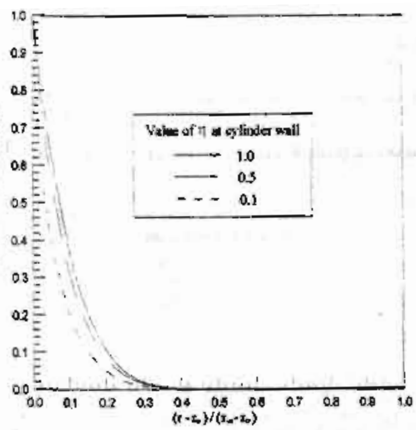

Figure (10) Dimensionless temperature distribution at different axial positions

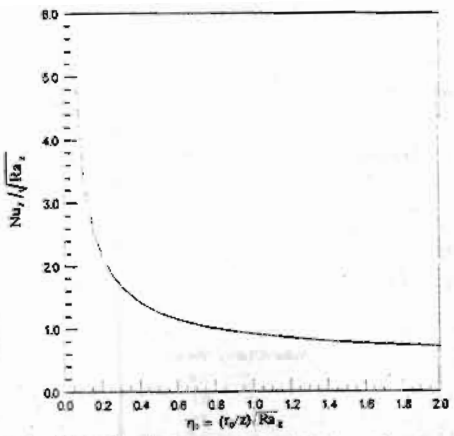

Figure (11) Local Nusselt number versus the local similarity variable at the wall

Figure (12) shows local Nusselt number along the cylinder length at different values of Rayleigh number Ra_o. Local Nusselt number Nu_o has its maximum value near the bottom-end of the cylinder, then it decreases slowly till it reaches its smallest value far from this end. At the same position, Nusselt number increases as Rayleigh number increases. Table (1) shows numerical values of local Nusselt number at different values of Rayleigh number.

Table (1) Local Nusselt number Nu_o for different values of Rayleigh number

			$Ra_o = 1.0$	$Ra_o = 5.0$	$Ra_o = 10.0$	$Ra_o = 20.0$
η_o	$\theta_\eta)_{wall}$	Nu_o	z / r_o	z / r_o	z / r_o	z / r_o
0.1	3.3965	0.3397	100	500	1000	2000
0.25	1.8729	0.4680	16	80	160	320
0.50	1.2678	0.6339	4.0	20	40	80
0.75	1.0395	0.7796	1.78	8.89	17.78	35.56
1.0	0.9165	0.9165	1.0	5.0	10	2.0
1.6	0.7673	1.2277	0.3391	1.9531	3.9063	10.2041
2.0	0.7141	1.4282	0.25	1.25	2.5	5.0

4. Conclusions

In this work the partial non-linear differential equations describing natural convection in porous medium from long cylinder are transformed to a set of first order differential equations, which are easier to be solved. Moreover, according to the present proposed local similarity solution, the characteristic physical quantities, affecting the natural convection in porous medium from long cylinder, are reduced to a single dimensionless quantity; Rayleigh number based on the cylinder radius Ra_o.

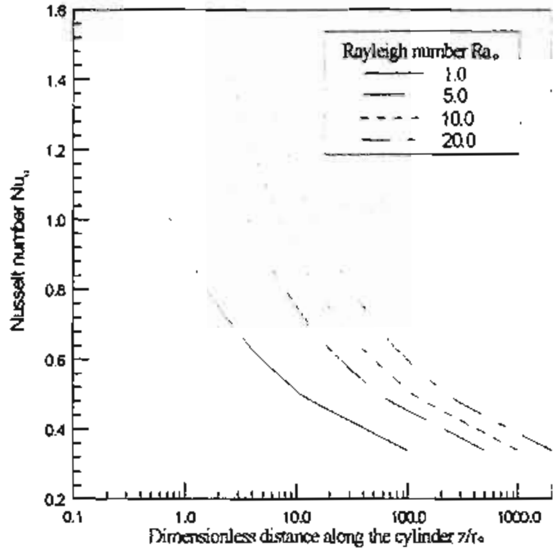

Figure (12) Local Nusselt number based on cylinder radius

Nomenclature

- f dimensionless stream function, $f(\eta) = \dfrac{\psi(r,z)}{\alpha \ z}$
- g gravitational acceleration
- K permeability of porous medium
- k thermal conductivity of fluid-saturated porous medium, $k = \phi \ k_f + (1-\phi) k_s$
- k_f thermal conductivity of fluid constituent
- k_s thermal conductivity of solid constituent
- p pressure
- Ra_o Rayleigh number base on cylinder radius, $Ra_o = K \ g \ \beta \ (T_o - T_\infty) \ r_o / \alpha \ \nu$

Ra_z Rayleigh number based on z, $Ra_z = K\ g\ \beta\ (T_o - T_\infty)\ z\ /\alpha\ \nu$
r, z radial and axial coordinates
r_o cylinder radius
T temperature of fluid-saturated porous medium
T_o temperature of cylinder surface
T_∞ temperature of porous medium far from the surface
V_r, V_z dimensionless radial and axial components of velocity
v_r, v_z radial and axial components of velocity

Greek symbols

α thermal diffusivity of homogenous porous medium, $\alpha = k\ /\rho_f\ c_{p_f}$

β coefficient of thermal expansion, $\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T})_p$

ϕ porosity of porous medium, ϕ = *pores volume / total volume*

η local similarity independent variable, $\eta = r\sqrt{\frac{K\ g\ \beta\ (T_o - T_\infty)}{\alpha\ \nu\ z}}$

μ, ν dynamic and kinematic viscosity
θ dimensionless temperature, $(T - T_\infty)/(T_o - T_\infty)$

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