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SUGGESTED APPROACHES FOR TRANSFORMING AN INTEGER PROGRAMMING PROBLEM INTO A KNAPSACK PROBLEM

N. EL-RAMLY

Menofiya University, Shebin-El-Kom, Menofiya, Egypt.

A. EL-KASSAS

Institute of National Planning, Nasr City, Cairo, Egypt.

ABSTRACT

On transforming to a Knapsack problem, an integer program (IP) can be seen to have a particularly simple structure . This allows us to develop two approaches for transforming an IP bounded variables into a Knapsack problem in bounded variables .

1. First Approach :

Consider the integer program :

Max cx, $x \epsilon S = \{x \mid Ax = b, O \leq x \leq U, x \text{ integer}\}$, U is an upper bound for x where A, b, c, and x are mxn, mx1, 1xn, 1xn matrices respectively .

Our intent here is to find weights $w = (w_1 , \dots , w_m)$, such that

$$T = \{x | wAx = wb \ , \ O \leq x \leq U \ , \ x \text{ integer} \} = S \ .$$

Since, T is described by one constraint

$$\sum_j \left(\sum_i w_i a_{ij} \right) x_j = \sum_j a_j x_j = b_o = \sum_i w_i b_i$$

It follows that an IP in bounded variables can be solved as an equality constraint Knapsack problem in bounded variables . We have to show that two constraints can be combined into one without changing the set of feasible solutions . Then, by repeating combining the constraints two at a time, it is clear that m constraints can be combined into one .

Consider the following two constraints :

$$\left.\begin{array}{l}\sum_{j=1}^{n} \alpha_j x_j - b_1 = 0 \\ \sum_{j=1}^{n} \beta_j x_j - b_2 = 0\end{array}\right\} \qquad (1)$$

and assume that the coefficients α_j and β_j , j = 1, ... , n are integers. Let

$$\lambda^+ = max \sum_{j=1}^{n} \alpha_j x_j - b_1 \ , \ \ O \leq x_j \leq U_j \ integer \ , \ \ j = 1, \ ..., \ n$$

$$\lambda^- = min \sum_{j=1}^{n} \alpha_j x_j - b_1 \ , \ \ O \leq x_j \leq U_j \ integer \ , \ \ j = 1, \ ..., \ n$$

Define $\alpha_j^+ = max\{O, \alpha_j\}$ and $\alpha_j^- = min\{O, \alpha_j\}$, we have $\lambda^+ = \sum_{j=1}^{n} \alpha_j^+ U_j - b_1$ and $\lambda^- = \sum_{j=1}^{n} \alpha_j^- U_j - b_1$. Let $\lambda = max \ \{\lambda^+, \ |-\lambda|\}$

Lemma 1.1 :

The integer vector x^o, $O \leq x^o \leq U$ is a solution to (1) if and only if $\sum_{j=1}^{n} (\alpha_j + k\beta_j) x_j^o - b_1 - kb_2 = 0$, where k is any integer satisfying $|k| > \lambda$.

2. Second Approach :

In this method, we show how to transform an IP problem (with a system of linear equations) to a single linear equation problem in the following theorem.

Lemma 2.1 : (Glover[2])

Consider a system of two equations :

$$\left.\begin{array}{l}s_1 \equiv \sum_{j=1}^{n} a_{1j} x_j = b_1 \\ s_2 \equiv \sum_{j=1}^{n} a_{2j} x_j = b_2\end{array}\right\} \qquad (2)$$

where all coefficient a_{ij} , b_i are integers, and at least one of b_1 and b_2 is not zero. Let w_1 and w_2 be relatively prime (nonzero) integers . If there exists at least one nonnegative integer solution to (2), then every nonnegative integer solution to

$$w_1 s_1 + w_2 s_2 = w_1 b_1 + w_2 b_2 \tag{3}$$

is a nonnegative integer solution to (2), and conversely, provided that

$$w_1 a_{1j} + w_2 a_{2j} \geq |b_2 a_{1j} - b_1 a_{2j}| \tag{4}$$

for j = 1, ..., n and (4) holds as a strict inequality for $j \epsilon J$, where J is any nonempty subset of {1, ... , n} such that all nonnegative solutions to (3) satisfy $x_j > 0$ for at least one j in J .

The lemma implies that w_1 and w_2 be chosen so that $w_1 b_1 + w_2 b_2 > 0$. Also by (4) the coefficient in (3) (i.e. $w_1 a_{1j} + w_2 a_{2j}$, $j = 1, ..., n$) are nonnegative . Thus, every nonnegative integer solution to (3) must have $x_j > 0$ for at least one j where its coefficient in (3), $w_1 a_{1j} + w_2 a_{2j}$ is positive . Consequently, the set J can consist of those j for which $w_1 a_{1j} + w_2 a_{2j} > 0$.

3. Example :

Consider the following integer programming problem :

$$Max\ Z = x_1 + x_2 + x_3$$

subject to

$$2x_1 + x_2 + x_3 + x_4 = 6 , \tag{5}$$

$$x_1 + 2x_2 + x_3 + x_5 = 6 , \tag{6}$$

$$x_1 + x_2 + 2x_3 + x_6 = 6 , \tag{7}$$

$$x_1 + x_2 + x_3 + x_7 = 4 , \tag{8}$$

and

$$x_1 \leq 3,\ x_2 \leq 3,\ x_3 \leq 3,\ x_4 \leq 6,\ x_5 \leq 6,\ x_6 \leq 6,\ and\ x_7 \leq 4,$$

$$x_j \geq 0\ integers,\ \ j = 1, \ldots, 7\,.$$

Method 1.

First to solve our example by the first approach,

For equation (8),

$$\left.\begin{array}{l}\lambda^+ = 1(3) + 1(3) + 1(3) + 1(4) - 4 = 9 \\ \lambda^- = 0(3) + 0(3) + 0(3) + 0(4) - 4 = -4\end{array}\right\} \Longrightarrow \lambda = 9 \Longrightarrow k = 10.$$

Combining with eq. (7) we have,

$$x_1 + x_2 + x_3 + x_7 + 10(x_1 + x_2 + 2x_3 + x_6) = 4 + 6(10) \Longrightarrow$$

$$11x_1 + 11x_2 + 21x_3 + 10x_6 + x_7 = 64 \tag{9}$$

For equation (6),

$$\lambda^+ = 12,\ \lambda^- = -6 \Longrightarrow \lambda = 12 \Longrightarrow k = 13\,.$$

Combining with eq. (9) we have,

$$144x_1 + 145x_2 + 274x_3 + x_5 + 130x_6 + 13x_7 = 838 \tag{10}$$

For equation (5),

$$\lambda^+ = 12,\ \lambda^- = -6 \Longrightarrow \lambda = 12 \Longrightarrow k = 13\,.$$

Combining with eq. (10) we have,

$$1874x_1 + 1886x_2 + 3563x_3 + x_4 + 13x_5 + 1690x_6 + 169x_7 = 10900$$

Thus the corresponding Knapsack problem is in the form,

$$Max \qquad x_1 + x_2 + x_3$$

$$s.t.\ 1874x_1 + 1886x_2 + 3563x_3 + x_4 + 13x_5 + 1690x_6 + 169x_7 = 10900$$

and $x_j \geq 0$ and integers, j = 1, ..., 7 ,

$$x_1 \leq 3,\ x_2 \leq 3,\ x_3 \leq 3,\ x_4 \leq 6,\ x_5 \leq 6,\ x_6 \leq 6,\ and\ x_7 \leq 4\ .$$

Method 2.

Now, our example can be solved by the second method as follows :

The initial system is :

var. / Eq.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	
(5)	2	1	1	1	0	0	0	6	×
(6)	1	2	1	0	1	0	0	6	
(7)	1	1	2	0	0	1	0	6	
(8)	1	1	1	0	0	0	1	4	

To combine equations (5) and (6), let J = {1, ..., 5} and by (4) we have:

J	$\|b_2a_{1j} - b_1a_{2j}\|$	Equation ~~(4)~~
1	6	$2w_1 + w_2 > 6$
2	6	$w_1 + 2w_2 > 6$
3	0	$w_1 + w_2 > 0$
4	6	$w_1 > 6$
5	6	$w_2 > 6$

Let $w_1 = 7$, and $w_2 = 8$, we have,

var. / Eq.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	
(11)	22	23	15	7	8	0	0	90	×
(7)	1	1	2	0	0	1	0	6	
(8)	1	1	1	0	0	0	1	4	

To combine (11) and (7), let J = {1, 2, ..., 6} and again use (4)

we obtain:

J	$\lvert b_2a_{1j} - b_1a_{2j}\rvert$	Equation (4)
1	42	$22w_1 + w_2 > 42$
2	48	$23w_1 + w_2 > 48$
3	90	$15w_1 + 2w_2 > 90$
4	42	$7w_1 > 42 \Longrightarrow w_1 > 6$
5	48	$8w_1 > 48 \Longrightarrow w_1 > 6$
6	90	$w_2 > 90$

Taking $w_1 = 7$, and $w_2 = 91$, we have:

var. / Eq.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
(12)	245	252	287	49	56	91	0	1176
(8)	1	1	1	0	0	0	1	4

To combine (12) and (8), let J = {1, 2, ..., 7} and again use (4) we obtain:

J	$\lvert b_2a_{1j} - b_1a_{2j}\rvert$	Equation (4)
1	196	$245w_1 + w_2 > 196$
2	168	$252w_1 + w_2 > 168$
3	28	$287w_1 + w_2 > 28$
4	196	$49w_1 > 196 \Longrightarrow w_1 > 4$
5	224	$56w_1 > 224 \Longrightarrow w_1 > 4$
6	364	$91w_1 > 364 \Longrightarrow w_1 > 4$
7	1176	$w_2 > 1176$

Take $w_1 = 5$, and $w_2 = 1177$.

Thus the corresponding Knapsack problem is in the form,

$$Max \qquad x_1 + x_2 + x_3$$

$$s.t.\ 2402x_1 + 3437x_2 + 2612x_3 + 245x_4 + 280x_5 + 455x_6 + 1177x_7 = 10588$$

$x_j \geq 0$, and integers, j = 1, ..., 7 , and

$$x_j \leq 3\ ,\ j = 1,2,3\ ,\ \ x_j \leq 6\ ,\ \ j = 4,5,6\ ,\ \ x_7 \leq 4\ .$$

4. Conclusion :

From the previous suggested two approaches, the IP may be transformed into a Knapsack problem . The deduced problem can be solved as an ordinary Knapsack problem . The transformation is worthwhile only if the deduced Knapsack problem is easier to solve than the original IP . The drowback of this approach is that, the coefficients result from the aggregation process may appear relatively large .

REFERENCES

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طرق مقترحة لتحويل مسألة البرمجة الصحيحة الى مسألة نابساك Knapsack

عند تحويل مسألة البرمجة الصحيحة الى مسألة نابساك فإن هذا التحويل له هيكل خصا وبسيط. وهذا يسمح لنا بإستنباط طريقتين لتحويل البرمجة الصحيحة ذات متغيرات محددة.

والطريقة الأولى تنبنى على إيجاد ترجيحات بحيث ينتج أن مسألة البرمجة الصحيحة ذات المتغيرات المحددة يمكن حلها كمسألة نابساك بقيد واحد فى صورة معادلة وذات المتغيرات المحددة. وذلكم بدمج كل قيدين فى قيد وحداد بدون تغيير مجموعة الحلول المسموح بها. وبتكرار دمج القيود اثنين اثنين يتضح أن m قيداً يمكن دمجهم جميعاً فى قيد واحد.

أما الطريقة الثانية فتنبنى على تحويل مسألة البرمجة الصحيحة ذات نظام معادلات خطية الى مسألة ذات معادلة خطية منفردة .